Construction of the Inverse Matrix to a Block Upper Triangular Matrix

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***Abstract*—A new analytical method of constructing the inverse matrix to the check matrix of the system of embedded inner codes for the generalized error-locating codes is proposed.**

I. Introduction

The generalized error-locating codes (GEL codes) [1] are a promising class of high-rate codes which can be applied in modern information transmission systems. The encoding rule for a GEL code is

$$C = H^{-1}S,$$

where C is a codeword (code matrix) of the GEL code, H is the check matrix of the system of embedded inner codes, S is a codeword of outer codes. The check matrix H is

$$H = \begin{bmatrix} I & H_{1,2} & H_{1,3} & \vdots & H_{1,n-1} & H_{1,n} \\ 0 & I & H_{2,3} & \vdots & H_{2,n-1} & H_{2,n} \\ 0 & 0 & I & \vdots & H_{3,n-1} & H_{3,n} \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \vdots & I & H_{n-1,n} \\ 0 & 0 & 0 & \vdots & 0 & I \end{bmatrix},$$

where I are identity matrices of appropriate dimensions, $(n-1)$ is the number of inner codes. The code is defined over the characteristic two finite field.

The aim of this paper is the analytical determination of the inverse matrix to the check matrix H.

II. The construction of the inverse matrix

There is a well-known property (for example, [2]) that the inverse matrix to a triangular matrix is also a triangular matrix.

Let us formulate the theorem for the general case, not necessarily over the characteristic two finite field.

Theorem 1: The inverse matrix $B = [B_{i,j}]$ to the check matrix H is

$$B = \begin{bmatrix} I & B_{1,2} & B_{1,3} & \vdots & B_{1,n-1} & B_{1,n} \\ 0 & I & B_{2,3} & \vdots & B_{2,n-1} & B_{2,n} \\ 0 & 0 & I & \vdots & B_{3,n-1} & B_{3,n} \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \vdots & I & B_{n-1,n} \\ 0 & 0 & 0 & \vdots & 0 & I \end{bmatrix},$$

where blocks of the matrix B have appropriate dimensions and the form

$$B_{i,i+p} = \\ = \sum_{i<q_1<q_2<\ldots<q_m<i+p} (-1)^{m+1} H_{i,q_1} H_{q_1,q_2} \ldots H_{q_m,i+p} \quad (1)$$

for $i = 1, \ldots, n-1$; $p = 1, \ldots, n-i$.

Proof: Consider the formula (1). The sum is calculated for all sequences $i < q_1 < q_2 < \ldots < q_m < i+p$, including the trivial shortest sequence $i < i+p$, which corresponds to the term $-H_{i,i+p}$. Let $E = HB$. Blocks of the matrix E are

$$E_{i,i+p} = B_{i,i+p} + H_{i,i+1}B_{i+1,i+p} + H_{i,i+2}B_{i+2,i+p} + \ldots \\ + H_{i,i+p-1}B_{i+p-1,i+p} + H_{i,i+p}. \quad (2)$$

Taking into account that the matrix E is the identity matrix, we obtain the system of equations

$$E_{i,i+p} = 0 \text{ for } i = 1, \ldots, n-1;\ p = 1, \ldots, n-i. \quad (3)$$

We solve the system of equations (3) by induction, starting from $p = 1$, and prove formula (1). For $p = 1$ from (2) it follows that $E_{i,i+1} = H_{i,i+1} + B_{i,i+1}$ and $B_{i,i+1} = -H_{i,i+1}$. Assume that formula (1) is true for every $B_{i,i+t}$, $t < p$. Combining (2) and (3) for $E_{i,i+p}$, we get

$$B_{i,i+p} = -H_{i,i+1}B_{i+1,i+p} - H_{i,i+2}B_{i+2,i+p} - \ldots \\ - H_{i,i+p-1}B_{i+p-1,i+p} - H_{i,i+p} = \\ = -\sum_{q=i+1,\ldots,i+p-1} H_{i,q}B_{q,i+p} - H_{i,i+p}. \quad (4)$$

Note that for all components $B_{j,j+q}$ on the right side of equality (4) we have $q < p$. We substitute the expression for

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$$
\begin{aligned}
B_{i,i+p} = -\sum_{q=i+1,\ldots,i+p-1} H_{i,q} B_{q,i+p} - H_{i,i+p} = \\
= -\sum_{q=i+1,\ldots,i+p-1} \Big(H_{i,q} \sum_{q<q_2<\ldots<q_m<i+p} (-1)^m H_{q,q_2} H_{q_2,q_3} \ldots H_{q_m,i+p} \Big) - H_{i,i+p} = \\
= \sum_{q=i+1,\ldots,i+p-1} \Big(\sum_{i<q<q_2<\ldots<q_m<i+p} (-1)^{m+1} H_{i,q} H_{q,q_2} H_{q_2,q_3} \ldots H_{q_m,i+p} \Big) - H_{i,i+p} = \\
= \sum_{i<q_1<q_2<\ldots<q_m<i+p} (-1)^{m+1} H_{i,q_1} H_{q_1,q_2} \ldots H_{q_m,i+p}.
\end{aligned}
\tag{5}
$$

$B_{j,j+q}$ from (1) into (4). In this way we get all the necessary expressions for $B_{i,i+p}$. Fix some q, $i < q < i + p$, and consider all components in formula (1) with a fixed $q_1 = q$. This sum can be written as follows: (see formula (5) at the top of the page). The previous calculations are based on the fact that all sequences $i < q_1 < q_2 < \ldots < q_m < i + p$ (without the trivial sequence under $i < i + p$) can be represented as a concatenation of the sequence $i < q_1$ and all sequences $q_1 < q_2 < \ldots < q_m < i + p$. This completes the proof of Theorem 1. ■

III. Conclusion

The proposed method is useful in the analysis for the GEL codes encoding/decoding procedures.

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References

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