Lower Energy Bounds for Antipodal Spherical Codes and for Codes in Infinite Projective Spaces

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***Abstract*—We apply linear programming (polynomial) techniques for obtaining lower bounds for the potential energy of antipodal spherical codes. For codes attaining our bounds we prove Lloyd type theorems. We also provide general formulations of our recent universal lower bound on energy of codes in infinite projective spaces.**

I. Introduction

Let $C \subset \mathbb{S}^{n-1}$ be antipodal spherical code (i.e. $C = -C$) with inner products in $\{-1\} \cup [-s, s]$, where $s \in (0,1)$, $|C| = M$. For a given (extended real-valued) function $h : [-1,1] \to [0, +\infty]$, we consider the h*-energy* (or the potential energy) of C defined by

$$E(n, C; h) := \frac{1}{|C|} \sum_{x,y \in C, x \neq y} h(\langle x, y \rangle), \tag{1}$$

where $\langle x, y \rangle$ denotes the inner product of x and y. The potential function h is called k*-absolutely monotone* on $[-1, 1)$ if its derivatives $h^{(i)}(t)$, $i = 0, 1, \dots, k$, are nonnegative for every $t \in [-1, 1)$. An antipodal spherical code $C \subset \mathbb{S}^{n-1}$ is called k-universally antipodal if it has minimum possible h-energy among all antipodal codes on $\mathbb{S}^{n-1}$ of the same cardinality $M = |C|$ and for all k-absolutely monotone functions on $[-1, 1)$.

In this paper use a general linear programming bound on energy of antipodal codes to propose good lower bounds for codes with a few distinct inner products. We investigate codes which attain our bounds for small values of k and prove Lloyd-type theorems. In the last section we briefly describe a general description of our recently obtained universal lower bounds (see [3]) on energy of codes in infinite projective spaces viewed as polynomial metric spaces [10].

II. Preliminaries

A. Gegenbauer polynomials and the linear programming framework

For fixed dimension n, the normalized Gegenbauer polynomials are defined by $P_0^{(n)}(t) := 1$, $P_1^{(n)}(t) := t$ and the three-term recurrence relation

$$(i+n-2)\, P_{i+1}^{(n)}(t) := (2i+n-2)\, t\, P_i^{(n)}(t) - i\, P_{i-1}^{(n)}(t) \text{ for } i \geq 1.$$

We note that $\{P_i^{(n)}(t)\}$ are orthogonal in $[-1, 1]$ with a weight $(1-t^2)^{(n-3)/2}$ and satisfy $P_i^{(n)}(1) = 1$ for all i and n. We have $P_i^{(n)}(t) = P_i^{((n-3)/2,(n-3)/2)}(t) / P_i^{((n-3)/2,(n-3)/2)}(1)$, where $P_i^{(\alpha,\beta)}(t)$ are the Jacobi polynomials in standard notation [11].

If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree r, then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as

$$f(t) = \sum_{i=0}^{r} f_i P_i^{(n)}(t). \tag{2}$$

We use the identity (see, for example, [7, Corollary 3.8], [8, Equation (1.7)], [9, Equation (1.20)])

$$|C| f(1) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^{r} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} Y_{ij}(x) \right)^2 \tag{3}$$

as a source of estimations by polynomial techniques. Here $C \subset \mathbb{S}^{n-1}$ is a spherical code, f is as in (2), $\{Y_{ij}(x) : j = 1, 2, \dots, r_i\}$ is an orthonormal basis of the space $\mathrm{Harm}(i)$ of homogeneous harmonic polynomials of degree i and $r_i = \dim \mathrm{Harm}(i)$.

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Theorem II.1. *Let $f(t) = \sum_{i=0}^{\deg(f)} f_i P_i^{(n)}(t)$ be a real polynomial such that*
(A1) *$f(t) \le h(t)$ for $t \in [-s, s]$, and*
(A2) *the Gegenbauer coefficients satisfy $f_i \ge 0$ for even i.*
Then

$$E(n, C; h) \ge M f_0 - f(1) - f(-1) + h(-1) \tag{4}$$

for every antipodal code $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ and inner products in $\{-1\} \cup [-s, s]$.

Proof. Using (3) and the conditions of the theorem we consecutively have

$$\begin{aligned} Mf(1) + ME(n, C; h) = \\ M(f(1) + h(-1)) + \sum_{\langle x,y\rangle \in [-s,s]} h(\langle x, y\rangle) \ge \\ M(f(1) + h(-1) - f(-1)) + \sum_{x,y \in C, x \ne y} f(\langle x, y\rangle) = \\ M(h(-1) - f(-1)) + |C|^2 f_0 + \\ \sum_{i=1}^{\deg(f)} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} Y_{ij}(x) \right)^2 \ge \\ M(Mf_0 + h(-1) - f(-1)), \end{aligned}$$

which implies (4). □

B. Distance distributions of antipodal spherical designs

Let $C \subset \mathbb{S}^{n-1}$ be a spherical code and let $x \in C$. The nonnegative integers $A_t(x)$, $t \in [-1, 1)$, are defined by $A_t(x) = |\{y \in C : \langle x, y\rangle = t\}|$. The system of numbers $\{A_t(x) : -1 \le t < 1\}$ is called the distance distribution of C with respect to x. If C is antipodal, we clearly have $A_{-1}(x) = -1$ and $A_t(x) = A_{-t}(x)$ for every $x \in C$ and $t \in (0, 1)$.

Further, the distance distributions of spherical designs of strength τ satisfy the following Vandermonde-type system of $\tau + 1$ linear equations [2, Theorem 2.1] (see also [7, Theorem 7.4], [1], [6, Chapter 14]).

$$\sum_{i=1}^{m} A_{s_i}(x) s_i^j = b_j |C| - 1, \ j = 0, 1, \ldots, \tau, \tag{5}$$

where $s_1, \ldots, s_m$ are all distinct inner products between x and the points of $C \setminus \{x\}$, $b_0 = 1$, $b_{2k} = \frac{(2k-1)!!}{n(n+2)\cdots(n+2k-2)}$ and $b_j = 0$ for odd j.

C. Derived spherical codes

Suitable parameter sets for k-A-universally optimal antipodal codes can be investigated by consideration of their derived codes [7, Theorem 8.2]. For a code $C \subset \mathbb{S}^{n-1}$, a point $x \in C$ and an inner product α, the derived code of C with respect to x and α is the set

$$C_\alpha(x) = \{y \in C : \langle x, y\rangle = \alpha\}$$

re-scaled on $\mathbb{S}^{n-2}$.

It is not difficult to see that the inner products of $C_\alpha(x)$ are in the set

$$I_{\alpha,x} = \left\{ \frac{\beta - \alpha^2}{1 - \alpha^2} : \beta \text{ is an inner product of } C \right\} \cap [-1, 1).$$

Moreover, if C is a spherical τ-design then $C_\alpha(x)$ is a spherical $(\tau - \ell + 1)$-design, where $\ell = |I_{\alpha,x}|$.

III. Antipodal codes with inner products -1 and $\pm s$

Assume that h is 4-absolute monotone and $s \in (0, 1)$. Consider the polynomial

$$f(t) = \sum_{i=0}^{3} a_i t^i = \sum_{i=0}^{3} f_i P_i^{(n)}(t)$$

which satisfies the interpolation conditions

$$f(\pm s) = h(\pm s), \quad f'(\pm s) = h'(\pm s).$$

This implies

$$\begin{aligned} a_0 &= \frac{2(h(s) + h(-s)) - s(h'(s) - h'(-s))}{4}, \\ a_2 &= \frac{h'(s) - h'(-s)}{4s}. \end{aligned} \tag{6}$$

It also follows that $f(t)$ satisfies (A1) (here we need the fourth derivative of h to be positive).

For (A2), the only condition that must be checked is $f_2 \ge 0$. Since

$$f_2 = \frac{a_2(n-1)}{n} = \frac{(n-1)(h'(s) - h'(-s))}{4ns} \ge 0$$

follows from $h''(t) \ge 0$, we conclude that $f(t)$ satisfies the conditions of Theorem II.1. Therefore $E(n, C; h) \ge Mf_0 - f(1) - f(-1) + h(-1)$ for every antipodal code $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ and inner products in $\{-1\} \cup [-s, s]$. If the equality is attained then the inner products of C belong to $\{-1, s, s\}$. Note that $f_2 > 0$ implies that C is a spherical 3-design.

We now calculate the bound of $f(t)$ and compare it to the energy of some known codes. First, we have the bound

$$\begin{aligned} Mf_0 - f(1) - f(-1) + h(-1) = \\ h(-1) + M\left(a_0 + \frac{a_2}{n}\right) - 2(a_0 + a_2) = \\ h(-1) + (M-2)a_0 + \frac{(M-2n)a_2}{n} = \\ h(-1) + \frac{(M-2)(2(h(s) + h(-s)) - s(h'(s) - h'(-s)))}{4} + \\ \frac{(M-2n)(h'(s) - h'(-s))}{4ns} \end{aligned} \tag{7}$$

where a_0 and a_2 are taken from (6).

Assume that $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ has inner products in -1, $-s$ and s. Then the distance distribution of C with respect to x in fact does not depend on x and is given

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by $A_{-1}(x) = 1$ and $A_{-s}(x) = A_s(x) = \frac{M-2}{2}$. Therefore the h-energy of C is

$$E(n, C; h) = h(-1) + \frac{(M-2)(h(-s)+h(s))}{2} = h(-1) + (M-2)(a_2 s^2 + a_0). \tag{8}$$

Equating (7) and (8) we obtain that C is 4-universally optimal if $\left(\frac{M}{n} - 2\right) a_2 = (M-2) a_2 s^2$, i.e. for $s = \sqrt{\frac{M-2n}{n(M-2)}}$. Notice the necessary condition $M \geq 2n$.

Theorem III.1. *The codes from Table 1 are 4-universally optimal.*

TABLE I
SOME 4-UNIVERSALLY OPTIMAL ANTIPODAL SPHERICAL CODES.

Dimension	Cardinality	Inner products
n	$2n+2$	$-1, \pm 1/n$
3	12	$-1, \pm 1/\sqrt{5}$
5	20	$-1, \pm 1/3$
6	32	$-1, \pm 1/3$
7	56	$-1, \pm 1/3$
10	32	$-1, \pm 1/5$
15	72	$-1, \pm 1/5$
21	56	$-1, \pm 1/9$
21	72	$-1, \pm 1/7$
21	252	$-1, \pm 1/5$
22	352	$-1, \pm 1/5$
23	552	$-1, \pm 1/5$
28	128	$-1, \pm 1/7$
35	240	$-1, \pm 1/7$
36	128	$-1, \pm 1/9$
43	688	$-1, \pm 1/7$
45	200	$-1, \pm 1/9$

Investigation of the distance distributions of the derived codes allows obtaining a Lloyd-type theorem.

Theorem III.2. *If an antipodal code $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ and inner products -1, $-s$ and s is 4-universally optimal, then $(n, M, s) = (n, 4n, 1/\sqrt{2n-1})$ or $s = \sqrt{\frac{M-2n}{n(M-2)}}$ is rational and $\frac{M}{4} + \frac{M-4n}{4sn}$ is positive integer.*

Proof. For fixed $x \in C$, the derived code $C_s(x)$ is a spherical 2-design of cardinality $|C_s(x)| = A_s(x) = \frac{M-2}{2}$, inner products $s_1 = \frac{s}{1+s}$ and $s_2 = -\frac{s}{1-s}$. The distance distribution of $C_s(x)$ with respect to $y \in C_s(x)$ satisfies the equations $A_{s_1}(y) + A_{s_2}(y) = \frac{M-2}{2} - 1$, $s_1 A_{s_1}(y) + s_2 A_{s_2}(y) = -1$, $s_1^2 A_{s_1}(y) + s_2^2 A_{s_2}(y) = \frac{M-2}{2(n-1)} - 1$. Using $s^2 = \frac{M-2n}{n(M-2)}$ we obtain $A_{s_1}(y) + A_{s_2}(y) = \frac{M-4}{2}$ and $A_{s_1}(y) - A_{s_2}(y) = \frac{M-4n}{2ns}$.

If s is irrational then $A_{s_1}(y) = A_{s_2}(y)$ and $M = 4n$, whence $s = \frac{1}{\sqrt{2n-1}}$.

If s is rational, then $A_{s_1}(y) = \frac{M(1+ns)-4n(1+s)}{4sn} = \frac{M}{4} + \frac{M-4n}{4sn} - 1$ must be nonnegative integer. $\square$

The next table shows all possibilities for 4-universally optimal codes which satisfy the conditions of Theorem III.2

TABLE II
SOME POSSIBILITIES FOR 4-UNIVERSALLY OPTIMAL ANTIPODAL SPHERICAL CODES.

Dimension	Cardinality	Inner products
$n \geq 7$, odd	$4n$	$-1, \pm 1/\sqrt{2n-1}$
19	152	$-1, \pm 1/5$
20	192	$-1, \pm 1/5$
37	296	$-1, \pm 1/7$
41	492	$-1, \pm 1/7$
42	576	$-1, \pm 1/7$
45	1080	$-1, \pm 1/7$
46	1472	$-1, \pm 1/7$
47	2256	$-1, \pm 1/7$

IV. ANTIPODAL CODES WITH INNER PRODUCTS $-1, \pm s$, AND 0

Assume now that h is 6-absolute monotone. Consider the polynomial

$$f(t) = \sum_{i=0}^{5} a_i t^i = \sum_{i=0}^{5} f_i P_i^{(n)}(t)$$

which satisfies

$$f(\pm s) = h(\pm s), \quad f'(\pm s) = h'(\pm s),$$
$$f(0) = h(0), \quad f'(0) = h'(0).$$

It satisfies (A1) (here we need h to be 6-absolute monotone) and the conditions to be satisfied for (A2) are $f_2 \geq 0$ and $f_4 \geq 0$.

We have

$$f_4 = \frac{a_4(n^2-1)}{(n+2)(n+4)} \geq 0 \iff a_4 \geq 0,$$
$$f_2 = \frac{(6a_4 + (n+4)a_2)(n-1)}{n(n+4)} \geq 0 \iff 6a_4 + (n+4)a_2 \geq 0.$$

The interpolation conditions give

$$a_0 = h(0),$$
$$a_2 = \frac{4(h(s)+h(-s)) - s(h'(s)-h'(-s)) - 8h(0)}{4s^2}, \tag{9}$$
$$a_4 = \frac{s(h'(s)-h'(-s)) - 2(h(s)+h(-s)) + 4h(0)}{4s^4}.$$

Denoting $F(s) = s(h'(s)-h'(-s)) - 2(h(s)+h(-s)) + 4h(0)$ we consecutively calculate $F'(s) = s(h''(s) + h''(-s)) - (h'(s) - h'(-s))$ and $F''(s) = s(h'''(s) - h'''(-s)) \geq 0$ since $h'''(s)$ is increasing. Thus $F'(s)$ is increasing for $s \geq 0$ and $F'(s) \geq F'(0) = 0$, whence $F(s)$ is increasing for $s \geq 0$ and $F(s) \geq F(0) = 0$. Therefore $a_4 \geq 0$ and $f_4 \geq 0$.

The condition $f_2 \geq 0 \iff 6a_4 + (n+4)a_2 \geq 0$ is equivalent to

$$(3 - (n+4)s^2)G(s) \geq 0,$$

where $G(s) = s(h'(s)-h'(-s)) - 4(h(s)+h(-s)) + 8h(0)$. Similarly to above, straightforward calculations show that $G(s) \geq 0$ (now we get the fourth derivative of $G(s)$ equal

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to $s(h^{(5)}(s) - h^{(5)}(-s))$ and here we need again h to be 6-absolute monotone). Hence the condition (A2) holds true for every $s \in \left(0, \sqrt{\frac{3}{n+4}}\,\right]$.

We now calculate the bound of $f(t)$ from Theorem II.1 as follows:

$$\begin{aligned} &Mf_0 - f(1) - f(-1) + h(-1) = \\ &h(-1) + M\left(a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)}\right) - 2(a_0 + a_2 + a_4) = \\ &h(-1) + (M-2)a_0 + \frac{(M-2n)a_2}{n} + \frac{(3M - 2n(n+2))a_4}{n(n+2)}, \end{aligned} \tag{10}$$

where the coefficients a_0, a_2 and a_4 are taken from (9).

Assume now that $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ has inner products in -1, $-s$, 0 and s. To achieve the above energy bound C must be a spherical 5-design if $f_2 > 0$ and $f_4 > 0$. Then the distance distribution of C with respect to x is determined from the equations

$$\begin{aligned} 2A_s(x) + A_0(x) &= M - 2, \\ 2A_s(x)s^2 &= \frac{M}{n} - 2, \\ 2A_s(x)s^4 &= \frac{3M}{n(n+2)} - 2. \end{aligned}$$

(recall that $A_{-1}(x) = 1$ and $A_{-s}(x) = A_s(x)$). Resolving this we obtain $s = \sqrt{\frac{3M-2n(n+2)}{(n+2)(M-2n)}}$,

$$A_s(x) = A_{-s}(x) = \frac{M-2n}{2ns^2} = \frac{(n+2)(M-2n)^2}{2n(3M-2n(n+2))},$$

$$A_0(x) = M - 2 - 2A_s(x) = \frac{2M(n-1)(M-n(n+1))}{n(3M-2n(n+2))}$$

(of course, $A_s(x)$ and $A_0(x)$ must be nonnegative integers; again the distance distribution does not depend on x). Thus the h-energy of C is given by

$$\begin{aligned} E(n,C;h) = h(-1) + \frac{(n+2)(M-2n)^2(h(-s)+h(s))}{2n(3M-2n(n+2))} &+ \\ \frac{2M(n-1)(M-n(n+1))h(0)}{n(3M-2n(n+2))} &= \\ h(-1) + \frac{(n+2)(M-2n)^2(a_0 + a_2s^2 + a_4s^4)}{n(3M-2n(n+2))} &+ \\ \frac{2M(n-1)(M-n(n+1))a_0}{n(3M-2n(n+2))}&. \end{aligned} \tag{11}$$

The expressions (10) for the bound of $f(t)$ and (11) for the energy of C are equated for determining suitable parameters for 6-universally optimality.

Theorem IV.1. *The codes from Table 2 are 6-universally optimal.*

Theorem IV.2. *If an antipodal code $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ and inner products -1, $-s$, 0 and s is 6-universally optimal, then: $s = \sqrt{\frac{3M-2n(n+2)}{(n+2)(M-2n)}}$ is rational and $A_s(x) = \frac{(n+2)(M-2n)^2}{2n(3M-2n(n+2))}$, $X = \frac{(M-4n)(1+s)}{4ns} + \frac{(n-1)M(M^2-8nM+4n^2(n+2))}{4n(3M-2n(n+2))^2}$ and $Y = \frac{(n-1)(M-n(n+1))M^2}{n(3M-2n(n+2))^2}$ are nonnegative integers.*

TABLE III
SOME 6-UNIVERSALLY OPTIMAL ANTIPODAL SPHERICAL CODES.

Dimension	Cardinality	Inner products
4	24	$-1, 0, \pm 1/2$
6	72	$-1, 0, \pm 1/2$
7	126	$-1, 0, \pm 1/2$
8	240	$-1, 0, \pm 1/2$
22	2816	$-1, 0, \pm 1/3$
23	4600	$-1, 0, \pm 1/3$

Proof. For fixed $x \in C$, the derived code $C_s(x)$ is a spherical 3-design of cardinality $|C_s(x)| = A_s(x) = \frac{M-2n}{2ns^2}$, inner products $s_1 = \frac{s}{1+s}$, $s_2 = -\frac{s^2}{1-s^2}$ and $s_3 = -\frac{s}{1-s}$. The distance distribution of $C_s(x)$ with respect to $y \in C_s(x)$ satisfies the system

$$\left|\begin{aligned} X + Y + Z &= \tfrac{M-2n}{2ns^2} - 1 \\ s_1X + s_2Y + s_3Z &= -1 \\ s_1^2X + s_2^2Y + s_3^2Z &= \tfrac{M-2n}{2n(n-1)s^2} - 1 \end{aligned}\right. \iff$$

$$\left|\begin{aligned} X + Y + Z &= \tfrac{M-2n}{2ns^2} - 1 \\ s(X - Z) &= \tfrac{M-2n}{2n} - 1 \\ X + Z &= \left(\tfrac{M-2n}{2n(n-1)s^2} - 1\right)\tfrac{(1-s^2)^2}{s^2} + \tfrac{M-6n}{2n} + s^2 \end{aligned}\right.$$

(here $A_{s_1}(y) = X$, $A_{s_2}(y) = Y$ and $A_{s_3}(y) = Z$ for short).

If s is irrational then $X = Z$ and $M = 4n$. This implies $s^2 = \frac{4-n}{n+2}$ which is possible only for $n = 3$ and leads to the icosahedron which does not have inner product 0.

If s is rational, then we have

$$\left|\begin{aligned} X &= \tfrac{(M-4n)(1+s)}{4ns} + \tfrac{(n-1)M(M^2-8nM+4n^2(n+2))}{4n(3M-2n(n+2))^2} \\ Y &= \tfrac{(n-1)(M-n(n+1))M^2}{n(3M-2n(n+2))^2} \\ Z &= X - \tfrac{M-4n}{2ns} \end{aligned}\right.$$

and X, Y, Z must be nonnegative integers. □

V. BOUNDS IN INFINITE PROJECTIVE SPACES

In a more general setting we describe universal linear programming bounds which are natural analogs of our bounds from [3]. The infinite projective spaces $\mathbb{R}P^{n-1}$, $\mathbb{C}P^{n-1}$, and $\mathbb{H}P^{n-1}$ are polynomial metric spaces [9], [10] are allow treatment which is very similar for the case of the Euclidean spheres $\mathbb{S}^{n-1}$.

Denote by $\mathbb{T}_\ell P^{n-1}$, $\ell = 1, 2, 4$, the projective spaces $\mathbb{R}P^{n-1}$, $\mathbb{C}P^{n-1}$, and $\mathbb{H}P^{n-1}$, respectively. We refer to [3] for conceptual explanations of the terms below (Levenshtein intervals, Levenshtein function, Delsarte-Goethals-Seidel numbers, Levenshtein $1/N$-quadrature nodes and weights).

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The Levenshtein intervals are

$$\mathcal{I}_m = \begin{cases} \left[t_{k-1,\ell}^{1,1}, t_{k,\ell}^{1,0}\right], & \text{if } m = 2k-1, \\ \left[t_{k,\ell}^{1,0}, t_{k,\ell}^{1,1}\right], & \text{if } m = 2k, \end{cases}$$

where $t_{i,\ell}^{a,b}$ is the greatest zero of the Jacobi polynomial $P_i^{(a+\frac{\ell(n-1)}{2}-1,b+\frac{\ell}{2}-1)}(t)$.

The Levenshtein function is given as

$$L(\ell,n,s) = \begin{cases} L_{2k-1}(\ell,n,s), & s \in \mathcal{I}_{2k-1} \\ L_{2k}(\ell,n,s), & s \in \mathcal{I}_{2k}. \end{cases}$$

where

$$L_{2k-1}(\ell,n,s) = \binom{k+\frac{\ell(n-1)}{2}-1}{k-1} \frac{\binom{k+\frac{\ell n}{2}-2}{k-1}}{\binom{k+\frac{\ell}{2}-2}{k-1}} \times \left[1 - \frac{P_{k-1}^{(\frac{\ell(n-1)}{2},\frac{\ell}{2}-1)}(s)}{P_k^{(\frac{\ell(n-1)}{2}-1,\frac{\ell}{2}-1)}(s)}\right]$$

and

$$L_{2k}(\ell,n,s) = \binom{k+\frac{\ell(n-1)}{2}-1}{k-1} \frac{\binom{k+\frac{\ell n}{2}-1}{k}}{\binom{k+\frac{\ell}{2}-1}{k}} \times \left[1 - \frac{P_{k-1}^{(\frac{\ell(n-1)}{2},\frac{\ell}{2})}(s)}{P_k^{(\frac{\ell(n-1)}{2}-1,\frac{\ell}{2})}(s)}\right].$$

The Delsarte-Goethals-Seidel numbers are:

$$D_\ell(n,\tau) = \begin{cases} \dfrac{\binom{k+\frac{\ell(n-1)}{2}}{k}\binom{k+\frac{\ell n}{2}-1}{k}}{\binom{k+\frac{\ell}{2}-1}{k}}, & \text{if } \tau = 2k-1, \\ \dfrac{\binom{k+\frac{\ell(n-1)}{2}}{k}\binom{k+\frac{\ell n}{2}}{k+1}}{\binom{k+\frac{\ell}{2}}{k+1}}, & \text{if } \tau = 2k. \end{cases}$$

The Levenshtein $1/N$-quadrature nodes $\{\alpha_{i,\ell}\}_{i=1}^k$ (respectively $\{\beta_{i,\ell}\}_{i=1}^k$), are the roots of the equation

$$P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0,$$

where $s = \alpha_{k,\ell}$ (respectively $s = \beta_{k,\ell}$) and $P_i(t) = P_i^{(\frac{\ell(n-1)}{2},\frac{\ell}{2}-1)}(t)$ (respectively $P_i(t) = P_i^{(\frac{\ell(n-1)}{2},\frac{\ell}{2})}(t)$) are Jacobi polynomials.

Theorem V.1. *Given the projective space $\mathbb{T}_\ell P^{n-1}$, $\ell = 1,2,4$, let h be a fixed absolutely monotone potential, n and N be fixed, and $\tau = \tau(n,N)$ be such that $N \in (D_\ell(n,\tau), D_\ell(n,\tau+1)]$. Then the Levenshtein nodes $\{\alpha_{i,\ell}\}$, respectively $\{\beta_{i,\ell}\}$, provide the bounds*

$$\mathcal{E}_\ell(n,N,h) \geq N^2 \sum_{i=1}^k \rho_i h(\alpha_{i,\ell}), \tag{12}$$

$$\mathcal{E}_\ell(n,N,h) \geq N^2 \sum_{i=0}^k \gamma_i h(\beta_{i,\ell}), \tag{13}$$

respectively, where $\mathcal{E}_\ell(n,N,h)$ is the infimum of the h-energy of codes in $\mathbb{T}_\ell P^{n-1}$ of cardinality N. The Hermite interpolants for $h(t)$ at these nodes are the optimal polynomials which solve the finite linear programming problem for polynomials of degree at most τ.

The bounds (12) and (13) are attained by codes which are called universally optimal [4]. In fact all known examples appear to be maximal (with respect to their cardinality) codes (see [9], [10], [5]).

Acknowledgment

The research of first author was supported, in part, by a Bulgarian NSF contract I01/0003. The research of second author was supported, in part, by a Simons Foundation grant no. 282207. The research of third and fourth authors was supported, in part, by the U. S. National Science Foundation under grants DMS-1412428 and DMS-1516400. The research of fifth author was supported, in part, by the Science Foundation of Sofia University under contract 57/2016.

References

[1] E. Bannai, N. J. A. Sloane, “Uniqueness of certain spherical codes”, *Can. J. Math.*, vol.33, 1981, pp. 437–449.

[2] P. Boyvalenkov, “Computing distance distribution of spherical designs”, *Lin. Alg. Appl.*, vol.226/228, 1995, pp. 277–286.

[3] P. Boyvalenkov, P. Dragnev, D. Hardin, E. Saff, M. Stoyanova, “Universal lower bounds for potential energy of spherical codes”, *Constructive Approximation*, DOI 10.1007/s00365-016-9327-5, online available: http://link.springer.com/article/10.1007/s00365-016-9327-5, in press, (arXiv:1503.07228).

[4] H. Cohn, A. Kumar, Universally optimal distribution of points on spheres, *J. Amer. Math. Soc.*, vol.20, 2006, pp. 99–148.

[5] H. Cohn, A. Kumar, G. Minton, Optimal simplices and codes in projective spaces, to appear in Geometry and Topology, arXiv:1308.3188.

[6] J. H. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. Springer – Verlag, New York, 1988.

[7] P. Delsarte, J.-M. Goethals, J. J. Seidel, “Spherical codes and designs”, *Geom. Dedicata,*, vol.6, 1977, pp. 363–388.

[8] V. I. Levenshtein, “Bounds for packings in metric spaces and certain applications”, *Probl. Kibernetiki*, vol.40, 1983 (in Russian), pp. 44–110.

[9] V. I. Levenshtein, “Designs as maximum codes in polynomial metric spaces”, *Acta Appl. Math.*, vol.25, 1992, pp. 1–82.

[10] V. I. Levenshtein, Universal bounds for codes and designs, Chapter 6 in *Handbook of Coding Theory*. V. S. Pless and W. C. Huffman, Eds., Elsevier, Amsterdam, 1998, pp. 499–648.

[11] G. Szegő, *Orthogonal Polynomials*. Amer. Math. Soc. Col. Publ., vol.23, Providence, RI, 1939.

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