Section 7

CIRCUIT THEORY

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Summary of the History of Circuit Theory*

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Summary—**After a brief survey of the state of circuit theory before World War I, the various directions of its development in the last 50 years are discussed, mainly in relation with applications to communication engineering. The early period of network design (1920–1925) was followed by the beginning of synthesis (1926–1935). The next steps were the development of feedback amplifier theory and insertion loss filter theory. The numerous new directions of research started during and after the second war are briefly mentioned. Finally recent progress is reported in formal realizability theory and in topological synthesis. A last section deals with nonlinear and linear variable circuits.**

Introduction

Although circuit theory is more than 100 years old (Ohm's law, 1827; Kirchoff's laws, 1845), it seems that no systematic account of its historical development has ever been written. The present essay attempts to cover the last 50 years, the fiftieth anniversary of the IRE being taken as an excuse to exclude the more distant past. This limitation is also justified by the development of circuit theory itself, which shifted from a steady to an accelerated progress a few years before World War I, simultaneously with the expansion of communication technology following the invention of the vacuum tube (de Forest's audion, 1906). This growth of circuit theory is directly testified by the number of articles published per year, which remained below unity till 1910 and increased from 5 to 25 in the period 1920–1940; after a drop during World War II, the increase continued, and a figure of 100 was exceeded in 1954.

We start with a brief survey of the state of the theory just before World War I and discuss in separate sections the various directions of its development up to the present days, mainly in relation with applications to communication engineering. Due to space restrictions, bibliographic references are omitted altogether: at least 200 important contributions out of a total of some 2000 would deserve mention. Important authors and dates are simply quoted in the main text; the dates generally refer to publications in regular scientific journals, for it was materially impossible to search through patents, theses and reports.

Circuit Analysis Before 1914

Long before 1914 circuit theory had emerged, from general electromagnetic theory, as an independent discipline with original concepts and methods. The conception of a circuit as a system of idealized lumped elements is already firmly established—drawings of Leyden jars and rheostats have gradually disappeared in favor of the now familiar graphical symbols. This assumes, at least implicitly, that a resistor is considered as a 2-terminal *black box* defined by the relation $v = Ri$, rather than as a physical device made of metal or carbon. This abstract point of view becomes

* Received by the IRE, May 22, 1961.

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more prominent in the modern developments, where the laws defining the elements, and the interconnection constraints (Kirchoff's laws) are explicitly taken as postulates. With this approach, *network theory*[1] only studies the properties of systems of elements and becomes a purely mathematical discipline dealing with abstract structures generated by various sets of postulates. Another consequence of this point of view is that the physical devices themselves are no longer studied by network theory but by what is sometimes called *device theory;* complicated physical devices are then naturally described by their *equivalent circuits.*

Conventional network elements are linear and time-independent; moreover, the stored electric and magnetic energies, and the dissipated power, are positive definite quadratic forms, and the constraints are instantaneously workless. This reduces the problem of network analysis to the classical theory of small vibrations in dissipative mechanical systems. The use of complex variables to combine the amplitudes and phases of harmonic steady states, and the separation of transients into normal modes, all familiar in analytical dynamics, were naturally taken over by circuit analysts. In particular, Steinmetz (1894) vulgarized the use of complex quantities in electrical engineering. Some specifically electric properties were, however, also established; we mention Kirchoff's topological rules (1847) and their extension by Feussner (1902–1904), the so-called Thévenin's theorem (Helmholtz, 1853), the star-delta transformation (Kennelly, 1899) and the concept of duality (Russel, 1904).

The impedance concept was fluently used by Heaviside, with p treated both as a differential operator and as an algebraic variable. The use of complex values for p is physically justified by Campbell in 1911. Although the relation between Heaviside's operational calculus and integration in the complex plane was only clarified after 1916, the mathematical tools allowing to derive transient behavior from harmonic response were available, since all is needed for lumped circuits is Heaviside's formula based on a partial fraction expansion. It took, however, 20 more years to introduce the terminology of poles and zeros into common engineering practice.

Although network analysis is, in principle, already separated from line theory, both branches remain in close contact in their later evolution, so that a few words about the latter are not out of place. Classical line theory has been worked out in detail by Heaviside, and such concepts as matching, reflection, iterative impedance, etc. began to be transferred from continuous to discrete structures, probably through the intermediary of artificial lines, a subject to which Kennelly devoted a book in 1917. Although the general concept of quadripole, or 2 port, as a black box characterized by its voltage equations, does not explicitly appear before 1921 (Breisig), attenuators and artificial lines calibrated in "equivalent length of standard line" were used earlier for telephone transmission measurements (the decibel was only introduced in 1924). Finally, Heaviside's invention of inductive loading successfully applied by Pupin led Campbell, through the theory of iterative structures (1903), to the invention of the electric filter to be discussed in the next section.

Network Design, 1920–1925

The progress in circuit theory during this period is connected with the development of long distance telephony in more than one respect. First, the theory of loaded lines is closely related with the invention of the electric filter, which found an immediate application in carrier telephony (first realiztion in 1918, Pittsburgh-Baltimore). Secondly, the design of bidirectional amplifiers (two-wire repeaters) immediately raised several problems of network synthesis (hybrid coils, balancing networks, filters, equalizers), not to mention the question of over-all stability. Although satisfactory engineering solutions were rapidly found to all these problems, their theoretical foundation was insufficiently systematic—the art of design, with its cut and try procedures had not yet matured into modern synthesis.

The electric filter was invented during World War I, independently by Campbell and Wagner. The first filters were iterative ladder stuctures, although lattice sections (in particular all-pass sections) are discussed by Campbell (1920–1922). After Zobel's invention of m-derivation (1923–1924), the catalogue of elementary sections was sufficiently extended to cover most practical requirements. In the spirit of line theory, a filter was designed as a cascade connection of sections with matched image-impedances, but the difference between image attenuation and insertion loss was thoroughly estimated by Zobel.

To Zobel are also due the understanding of the ideal transmission conditions (frequency independent attenuation and linear phase), the design of constant impedance equalizers (1928) including a method of approximation to prescribed frequency characteristics, the discovery of bandwidth conservation in the low-pass /band-pass transformation, and the first correction for mutual bridging effects in a parallel connection of two filters (x-derivation).

Simultaneously with the development of filter theory, the general concept of a quadripole, with its impedance, admittance and chain (ABCD) matrices was introduced in Germany and France, and the rules for computing the matrices of series, parallel and tandem combinations of quadripoles were discovered. One distinguished

[1] Our restriction of the term *network* (as opposed to *circuit*) to situations where reference is made to an idealized model conforms with the IRE Standards ("IRE Standards on Circuits: Definitions of Terms for Linear Passive Reciprocal Time Invariant Networks, 1960" (60 IRE 4.S2), Proc. IRE, vol. 48, p. 1609; September, 1960). Both terms are in fact synonymous in common usage. Another distinction has sometimes been made which reserves *network* for systems with free terminals or terminal-pairs (Campbell and Foster, 1920); in this terminology, an n-port becomes a circuit when terminations are specified at all ports.

between 2-terminal pair networks and 3-terminal networks (grounded quadripoles), and the combination rules were correctly restricted to cases where no "longitudinal voltage" appears. Here again, the separation of a general disturbance into a transversal and a longitudinal wave (using both conductors as "go circuit" and the earth as "return") was familiar in line theory, and effectively used in long distance telephony (phantom circuits) since 1903. A similar separation into three modes (symmetrical components) for three-phase power circuits was introduced by Fortescue in 1918.

The concept of n-terminal-pair network, or n-port, does not seem to have appeared during this period. It occurs, however, implicitly in a 1920 paper by Campbell and Foster, which is probably the first publication on network synthesis in the true sense—the energy relations in matched nondissipative 4 ports are discussed, the biconjugacy of the networks is proved, and a complete enumeration of all realizations is given; moreover, the circuits (which include the familiar hybrid coil) are explicitly treated as composed of ideal transformers, a new network element whose theory is established.

Acoustical telephone repeaters (telephone+microphone) were used before the invention of the vacuum tube, and difficulties due to singing were experienced. It is not easy to trace the evolution during the first war, and the post war publications already show a well-established technique. This is testified by the Campbell-Foster paper of 1920, by the design formulas for balancing networks (Hoyt, 1923–1924), and by the recognition of the fact that singing is essentially limited by line irregularities (Crisson, 1925).

The Beginning of Synthesis, 1926–1935

Although Foster's proof of his reactance theorem (1924) is already a transition from the methods of analytical dynamics to those of modern network synthesis, the first paper dealing explicitly with the realization of a one-port whose impedance is a prescribed function of frequency is Cauer's 1926 contribution, based on continuous fraction expansions (also studied by Fry, 1926). With Cauer's and Foster's theorems, the synthesis problem for one ports containing two kinds of elements only was solved. The analogous problem for general one ports was solved by Brune (1931) and led to the concept of positive real function. Pomey (1928) proved that the real part of the impedance matrix of a general passive 2-port was positive definite at real frequencies. The remaining developments of this period deal mainly with nondissipative n-ports; although Gewertz (1933) found a synthesis method for general 2-ports (containing all three kinds of elements), the general problem for n-ports was only solved after World War II and will be discussed in another section. The concentration on LC networks is closely connected with filter theory and its engineering interest. Another aspect of network synthesis, the approximation problem, made also its appearance during this period; the maximally flat approximation was used by Butterworth (1930) in the design of multistage amplifiers; simultaneously and independently, Cauer realized the optimal character of the Chebyshev approximation and solved the approximation problem for an important class of image-parameter filters. Finally, it should be remarked that the canonical structures obtained as solutions of the various synthesis problems made a free use of ideal transformers; the much more difficult problem of synthesis without transformers was not of paramount interest for communication applications and has only been treated recently.

The simplest network after the one port is the symmetric 2-port, which involves two frequency functions only. Geometrically symmetric 2-ports were treated by Bartlett (1927) and Brune (1932), whereas Cauer (1927) and Jaumann (1932) found a number of canonical circuits for all symmetric 2-ports. Dissymetric, and, in particular, antimetric 2-ports were studied by Cauer, who also extended Foster's theorem to LC n-ports (1931) and showed (1932–1934) that all equivalent LC networks could be derived from each other by the linear transformations considered by Howitt (1931). Certain classes of symmetric n-ports were studied by Baerwald (1931–1932).

Cauer's first book on filter design (1931) contains tables and curves for the Chebyshev approximation to a constant attenuation in the stop-band of an image parameter low-pass filter, as well as frequency transformations for other filter classes. The solution of the approximation problem involved rational functions whose extremal properties were established by Zolotareff in 1877 and which reduce to ordinary Chebyshev polynomials when elliptic functions are replaced by trigonometric functions. Cauer's presentation of his design data was based on canonic structures, practically less convenient than ladder structures, and was not accepted in engineering circles before it was realized that the statement and the solution of the approximation problem were of interest in themselves, for most of Cauer's results could easily be transferred to the ladder structure. The systematic theory of image parameter filters was further developed by Bode (1934) and Piloty (1937–1938), thus placing Zobel's earlier results in a clearer perspective. The particular problems raised by crystal filters were studied by W. P. Mason (1934–1937).

Feedback Amplifiers, 1932–1945

The construction of oscillators was one of the first applications of vacuum tubes, and the amplification increase due to a positive feedback below the threshold of oscillation is mentioned in several early publications on radio-engineering. The elementary, but erroneous, physical reasoning based on the round and round circulation of the signal in the feedback loop yielded the geometric series $\mu(1+\mu\beta+\mu^2\beta^2+\cdots)$ for the effective amplification. This gave the impression that instability must occur for all $|\mu\beta|>1$, for the series then diverges.

Stable behavior with a negative feedback exceeding 6 db was thus apparently forbidden, and such low values have little practical interest. It was soon recognized that the theory based on the geometric series was contradicted by experience, and the expression $\mu/(1-\mu\beta)$ was used even for $|\mu\beta|>1$ without theoretical justification until Nyquist (1932) proved his famous stability criterion and showed the error in the older theory—the physical reasoning based on the loop circulation is only correct when transients are taken into account, the terms in the geometric series then becoming convolution products; the steady-state formula is obtained as the limit after infinite time; the transient series is convergent, but not always uniformly, so that the limit operation and the summation cannot be interchanged. The case of conditional stability, predicted by Nyquist's theory, was experimentally confirmed (Peterson, *et al.* 1934) and the advantages of negative feedback were systematically discussed by Black (1934).

The domestication of negative feedback made it possible to design the wide-band highly linear amplifiers required for multichannel carrier telephony. Although various designs for interstage networks were proposed, for instance by Wheeler and Percival, the limitations due to parasitic capacities, and the way of overcoming them by an optimum over-all design, were only clarified by Bode in 1940, with the help of the integral relations between attenuation and phase.

The fact that the real and imaginary parts of physical network functions could not be specified independently from each other was first stated in connection with the ideal filter paradox. Küpfmüller's treatment (1926) of the transient response to a unit-step led to the well-known sine-integral embodying a response preceding the stimulus. Restrictions imposed by causality to physical response functions are mentioned by Y. W. Lee (1932) in connection with a method of synthesis for arbitrary transfer functions based on Fourier transforms, a method patented by Lee and Wiener (1938). Physical approximations to the ideal filter response, both in amplitude and phase, are discussed by Bode and Dietzold (1935). Explicit integral relations between real and imaginary parts of various network functions were studied independently by Cauer (1932–1940), Bayard (1935) and Leroy (1935–1937), but similar relations were known earlier in the theory of optical dispersion (Kramers-Kronig, 1926).

Bode extended the relations to the case where either component is specified in a partial frequency range, and worked out their consequences for input, output and interstage networks with prescribed parasitic capacities. He also computed the maximum obtainable over-all feedback in terms of band width and asymptotic loop transmission. The stability criterion for multi-loop amplifiers was obtained by Llewellyn, and the effect of feedback on impedances was discussed by Blackman (1943). Bode introduced the concepts of return reference and sensitivity in his classical book (1945).

Insertion Loss Filters and Related Problems

The limitations of image-parameter theory first appeared in connection with the design of filter groups, a problem frequently encountered in carrier telephony. Zobel's procedure of x-derivation, already mentioned, was first replaced by a more systematic method of impedance correction (Bode, 1930). An image-parameter theory of constant impedance filter pairs was developed by Brandt (1934–1936), Cauer (1934–1937) and Piloty (1937–1939), and it was recognized that this also yielded a solution to the equivalent problem of open-circuit filter design.

A completely new approach to the whole problem is contained in Norton's paper (1937) on constant impedance filter pairs, where the method of design starting from a prescribed insertion loss is established. The general synthesis problem for a reactance 2-port with prescribed insertion loss was solved independently by Cocci (1938–1940), Darlington (1939), Cauer (1939–1941) and Piloty (1939–1941). These contributions establish the canonic realization of a reactance 2-port as a ladder structure with mutual inductances restricted to adjacent or nearly adjacent arms and, as a consequence, the possibility of realizing an arbitrary passive one port by a network containing one resistance only. The approximation problem for insertion loss filters was reduced to the similar problem for image-parameter filters. Finally, Darlington also devised a method for precompensating the dissipative distortion.

As already mentioned, the prewar evolution of network theory was closely related with the development of wire communication, and the perfection reached by filter and feedback amplifier theory around 1940 made possible the design to strict tolerances which is required in long distance telephone equipment. On the contrary, the easier narrow-band problems of radio-engineering were treated by elementary circuit analysis, and it is only with the advent of video and pulse techniques that the theory of wide-band multi-stage amplifiers (without over-all feedback) underwent a systematic development, mainly during World War II. Another direction of war-time evolution was the extension of filter techniques to higher frequencies, leading to transmission line filters and microwave networks. Both directions have influenced classical filter theory and, as a consequence, common mathematical methods are now used in an extended field.

In microwave applications, the classical description of network performance in terms of voltages, currents, impedance and admittance matrices, was naturally replaced by a description based on transmitted and reflected wave amplitudes, leading to the concept of scattering matrix (Montgomery, Dicke and Purcell, 1948) taken over from general physics. This concept is also of interest in the field of lumped networks, where it was introduced independently and simultaneously by Belevitch. The scattering formalism allowed an easier presentation of insertion loss filter theory, and has been of great help in other applications to be discussed later.

The war-time progress in amplifier design is described in the book of Valley and Wallmann (1943), but similar work has been done independently in Germany, namely by Cauer (posthumous publications). Most input and inter-stage circuits are actually ladder filters, terminated or open-circuited, so that filter and amplifier problems are closely related. Explicit formulas for the element values of various important classes of ladder filters have been investigated by many workers, but it was recently realized that most results had been anticipated by Takahasi (paper in Japanese, 1951; English adaptation, 1960). The design of input and output circuits is also related with the broad-band matching problem, which consists in constructing a 2-port which transforms a given frequency-dependent output load (for instance a resistance shunted by a parasitic capacitance) into a pure input resistance. A rigorously constant resistance can be obtained by a lossy 2-port, but it is practically important to know the approximation to matching in a given frequency range obtainable from a lossless 2-port, and to see how the transmission loss varies with the degree of impedance equalization for lossy 2-ports. The broad-band matching problem with lossless 2-ports was solved by Fano (1950) for important classes of load impedances. The relations for lossy 2-ports were obtained by Carlin and La Rosa (1952–1955) with the help of the scattering formalism.

Distributed amplifiers, which overcame the limitations imposed by parasitic capacities to stagger-tuned circuits, had been invented by Percival in 1937, but were only practically exploited after 1948; the problem of their optimum design is still unsolved, although some progress has been recently achieved.

The scattering formalism proved useful in the design of various classes of n-ports of interest in telephone applications. Belevitch (1950–1955) discussed matched n-ports with equal losses between any couple of ports, and biconjugate n-ports for $n>4$. Dosoer (1958) and Oswald (1958) established design methods for a class of filter 4-ports (invented by Darlington in 1938) used in submerged repeaters.

The methods of best approximation used in filter and amplifier design were applied to other problems, namely to the design of various classes of delay networks. The maximally flat approximation for the group delay was obtained by Thomson (1949), and the Chebyshev approximation by Ulbricht and Piloty (1960). The Chebyshev approximation to a constant phase difference between two all-pass 2-ports (of interest for polyphase modulation) was obtained independently by Darlington, Orchard and Saraga (1950), and a mathematically equivalent problem on the phase angle of an impedance (of interest in feedback amplifier design) was solved by Baumann (1950). For problems having no exact analytic solution, Darlington (1952) described a procedure based on a series expansion in Chebyshev polynomials, whereas the potential analogy and the related electrolytic tank technique were practically applied at least since 1945 (Hansen and Lundstrom).

New Trends in Post-War Evolution

The field of application of circuit theory extended in so many new directions during and after World War II that only the major tendencies can be outlined. Several new developments lie on the borderline between circuit theory and other disciplines (information theory and noisy systems, electronic computers, automatic control), and will not be discussed. The present section attempts a brief classification of the new ideas, concepts and methods, in relation with their engineering applications; the next two sections review in detail the postwar developments of network theory *stricto sensu*. Nonlinear and linear variable circuits are treated separately in the last section.

Pulse techniques, already mentioned, raised various synthesis problems in the time domain. Although every problem stated in the time domain is, in principle, equivalent to a problem in the complex frequency domain, the approximation requirements are often difficult to translate from one domain into the other, and the convergence properties in the two domains may be quite different. Various mathematical methods, such as Laguerre functions (Lee, 1932), time series expansions (Lewis, 1952), complex integration (Guillemin and Cerillo, 1952), have been used, but there seems to have been little fundamental progress; for instance, the problem of the steepest monotonic response in presence of a given parasitic capacity is not completely solved, although important contributions have been published, namely by Zemanian (1954) and Papoulis (1958).

Microwave circuits have also been previously mentioned. The realization of a microwave gyrator (Hogan, 1952), based on the Faraday effect in ferrites, justified the interest of the ideal gyrator concept (Tellegen, 1948) and promoted theoretical research on nonreciprocal networks initiated by Tellegen (1948–1949). Theoretical work on the synthesis of passive n-ports (both reciprocal or not) is discussed in the next section. Its practical importance is due to the fact that, without the limitations of reciprocity, better performances can be obtained (interstage networks; Tellegen, 1951) or otherwise impossible behavior can be achieved (design of circulators: Carlin, 1955).

Progress has been slower in the theory of active networks because it was more difficult to represent the real physical devices by simple ideal elements; needless to say, even greater difficulties await the circuit theorists of the future, with the advent of integrated microelectronic devices. Classical amplifier theory dealt in fact with passive networks separated by ideal unilateral buffer stages, and amplifiers with large negative feedback were used to realize the active transfer function $-1/\beta$ involving only the transfer ratio β of the passive feedback network. With the advent of the transistor

(Bardeen and Brattain, 1948), practical design problems became much more difficult due to the internal feedback of the device. The fundamental limitations of active non-unilateral devices, at a fixed frequency, were progressively understood; for instance, the conditions for intrinsic stability were established by Llewellyn (1952), power invariants were found by S. J. Mason (1954), and noise invariants by Haus and Adler (1956–1959). The analysis of complex feedback structures was clarified by the use of the flow-graph notation (S. J. Mason, 1953). After the introduction of gyrators, it appeared more convenient to represent active devices by an equivalent circuit containing gyrators and (positive and negative) resistors, so that negative resistance remained the only new element in the theory of active networks. Although various negative resistance effects (arc discharge, dynatron, etc.) had been known for many years, practical and economic devices simulating linear negative resistors became available only with solid-state components (negative impedance converter: Linvill, 1953; tunnel diode: Esaki, 1958), and this favored the adoption of negative resistance as the basic element in theoretical work. Recent results in the theoretical field are summarized at the end of the next section.

Returning now to the synthesis of passive reciprocal networks, we briefly comment on the progressive separation between formal realizability theory (where ideal transformers are freely accepted) and topological synthesis (where even mutual inductances are excluded). The recent interest in topological methods originated from several distinct fields. First, mutual inductances, and even self-inductances, are difficult to realize at very low frequencies; this stimulated research on RC circuits, mainly for servomechanism applications, although similar problems arose earlier in the design of RC-oscillators (invented by van der Pol and van der Mark in 1934). Secondly, even at higher frequencies, it may be economical to replace inductances by capacitances combined with positive and negative resistances; this possibility was known theoretically before 1930, but became only practical with the availability of solid-state devices. Topological problems also arose in the design of contact networks, and some recent developments in the theory of switching are related with various fundamental problems of conventional network theory. Finally, the treatment of network problems on electronic computers asked for a more complete and detailed algebraization of all topological notions and raised various enumeration problems. Topological analysis was recently discussed by many workers, such as Bryant, Okada, Percival, Seshu and Watanabe; for combinatorial and enumeration problems, we refer to the book of Riordan (1958).

Formal Realizability

The synthesis of passive reciprocal n-ports has been achieved by three methods. The first one extended Gewertz's procedure for 2-ports and consisted in the successive extraction of reduced impedance matrices; the process is heavy and laborious, and only of historical interest; it enabled, however, Oono (1946) and Bayard (1949) independently to prove that any positive real impedance or admittance matrix is realizable, thus showing that RLC elements and the ideal transformer constitute a *complete* system of passive reciprocal elements. After a discussion of the 2-port case by Leroy and Belevitch (1949), it appeared that the first method was actually a disguised extension of Darlington's process for one ports, *i.e.*, the realization of an n-port as a reactive $2n$-port closed on n resistors. The second method consists in a direct application of this idea; it was used by Bayard (1950) and with the scattering formalism by Belevitch (1951). The third method extends Brune's process for one ports and arrives to a structure containing the minimum number of reactive elements; synthesis by this method was achieved independently by Oono (1948), Mac Millan (1948–1952) and Tellegen (1953) and contributed to a satisfactory definition of the degree of a rational matrix. In an important paper (1954), Oono and Yasuura rediscuss the synthesis by the second method, using the scattering matrix both for reciprocal and nonreciprocal n-ports and solve completely the equivalence problem.

In connection with the design of narrow-band unsymmetrical band-pass filters, Baum (1957–58) introduced a new fictitious passive element, the imaginary resistance (or frequency-independent reactance), and showed its interest in various synthesis problems. Using this concept, Belevitch obtained a simple derivation of Brune's for one-ports (1959) and extended Tellegen's process to nonreciprocal n-ports (1960).

Extensions to active networks were delayed due to difficulties arising with certain pathological n-ports admitting none of the conventional matrix descriptions (Tellegen, 1954; Carlin, 1955). These difficulties were circumscribed by Oono (1960), and by Carlin and Youla (1960). The latter showed that any active n-port is realizable as a passive nondissipative $3n$-port closed on n positive and n negative resistors. In a companion paper, Youla and Smilen (1960) establish the gain-bandwidth limitations due to parasitic capacitances in negative-resistance amplifiers and derive design formulas for optimum amplifiers.

Topological Synthesis

Logically, but not historically, the first problem arising in this field is the one of discriminating between constraints which are realizable without ideal transformers and general workless reciprocal constraints. In synthesis problems, constraints generally appear under the form of prescribed incidence relations between loops and branches, and the problem is to determine the necessary and sufficient conditions under which a prescribed incidence matrix corresponds to a graph. A necessary condition is well known (the matrix must be totally unimodular, *i.e.*, have all its minors of all orders equal to 0 or ± 1), but is certainly not sufficient, a classical counter-example being the dual of the constraints in a nonplanar graph (Foster, 1952). Sufficient conditions were found by Tutte (1958–1959). Various algorithms have recently been devised which either prove a given matrix to be unrealizable, or lead to a unique realization, within trivial isomorphisms. The first such algorithm is due to Gould (1958) and was applied to the synthesis of contact networks.

The next problem is the synthesis of resistance n-ports, and of resistance networks having $n+1$ terminals (grounded n-ports). It was known for some time that dominancy (any diagonal element not smaller than the sum of the moduli of all elements in the same row) was sufficient, but not necessary, for the admittance matrix of an n-port. Necessary and sufficient conditions for a 3-port with prescribed impedance or admittance matrix were obtained by Tellegen (1952) by expressing that the network cannot yield current nor voltage gain under any set of open- or short-circuit conditions. A nontrivial extension of Tellegen's approach led Cederbaum (1958) to the condition of paramountcy (any principal minor not smaller than the modulus of every minor based on the same rows), which is weaker than dominancy. Paramountcy is necessary for admittance and impedance matrices, but its sufficiency is not established for impedance matrices with $n>3$. For $(n+1)$-terminal networks, the synthesis is equivalent to to a congruence transformation into a positive diagonal matrix by means of a realizable constraint matrix. An algorithm yielding a unique (except for trivial variants) transformation, if possible at all, has been found by Cederbaum, but this algorithm accepts totally unimodular transformation matrices, which are not necessarily realizable. Related synthesis procedures were described by Guillemin (1960) and Biorci (1961).

For two-element kind one ports, the canonical structures of Foster and Cauer contain no transformers. It has long been thought that the transformers appearing in Brune's process are unavoidable in the synthesis of general three-element kind one ports; a canonical realization without mutual inductance was, however, published by Bott and Duffin (1949). In spite of various small improvements (Pantell, 1954; Reza, 1954), the method is quite wasteful in elements; no further general improvement has been obtained, but procedures due to Miyata (1952) and Guillemin (1955) sometimes yield more economical realizations.

The synthesis of RC 2-ports was first treated by Guillemin (1949), who showed that the class of grounded RC 2-ports was practically not narrower than the class of general reciprocal 2-ports, in the sense that the modulus of the input-output voltage ratio of any 2-port of the second class can be arbitrarily approximated within the first class, except for a constant multiplier. Guillemin's synthesis was in terms of parallel ladders; Orchard (1951) established a synthesis by RC lattices and discussed the extraction of terminal resistances in order to realize prescribed insertion loss functions. The approximation problem was treated by Ozaki and Fujisawa (1953). Miscellaneous synthesis procedures were discussed by a number of authors, but the most general and complete results are due to Fialkow and Gerst in a half-dozen papers published between 1951 and 1955. These authors found the necessary and sufficient conditions for voltage ratios of grounded and nongrounded RC and RLC 2-ports, as well as for various restricted structures (ladder, lattice); they indicated canonic realizations and discussed the value of the constant multiplier which fixes the maximum available voltage gain. Articles by Cederbaum (1956) and Kuh and Paige (1959) bring certain additional precisions.

The synthesis of RC grounded 2-ports with prescribed admittance or impedance matrices is still an unsolved problem. A series-parallel synthesis procedure was described by Ozaki (1953), and sufficient conditions under which it succeeds are known, but have not been proved necessary. The problem progressed through contributions of Lucal (1955), Slepian and Weinberg (1958) and Adams (1958), but Darlington's conjecture (1955), stating that any RC grounded 2-port admits an equivalent series-parallel realization, is still unproved.

The problem of ladder synthesis of two-element kind 2-ports arose earlier in filter theory, and the important method of zero-shifting in cascade synthesis was introduced by Bader (1942). Necessary and sufficient conditions for ladder realizability, in the case of a prescribed impedance matrix, or prescribed insertion loss, are unknown in general, but have been established for important classes of networks (Fujisawa, 1955; Watanabe, 1958) and applied to filter design (Meinguet, 1958).

Active grounded RC networks have recently been discussed by Kinariwala (1959–1960) and Sandberg (1960).

Nonlinear and Linear Variable Circuits

In contrast with the maturity of linear network theory, it is often considered that the theory of nonlinear and of linear variable networks is still in its infancy: it is not yet completely separated from nonlinear mechanics and has not reached the stage of synthesis. The present brief review limited to purely electrical problems is intended to show, however, that there has been some systematic progress and that the theory in its present state is no longer a collection of odd results.

First, it is important to separate the problems raised by the unavoidable nonlinearities appearing in nominally linear systems, and the intentional nonlinear effects allowing performances unobtainable with linear systems. The calculation of nonlinear distortion is relatively elementary in the case of slightly nonlinear characteristics (vacuum tube amplifiers, carbon microphone, etc). A more difficult problem arose in connection with the cross-modulation due to hysteresis in loading coils (Kalb and Bennett, 1935). The computation of the modulation products generated by two harmonic signals turned out to be mathematically equivalent to the similar problem in an ideal rectifier, and the amplitudes were obtained as hypergeometric functions. Further progress is reported in later publications by Bennett and others, and the results proved useful in a number of problems involving sharp nonlinearities, such as overload effects in rectifier modulators and feedback amplifiers.

The typically useful properties of nonlinear, or linear variable systems are related with frequency conversion in a wide sense, including harmonic or subharmonic generation, and even simple oscillation, for an oscillator transforms dc power into power at its own frequency. The theory of the triode oscillator was developed by Appleton and van der Pol (1920–1926), and explained such effects as synchronization, amplitude hysteresis, etc. Subharmonic generations in the triode oscillator were analyzed by Mandelstam and Papalexi (1931). Although van der Pol had shown the continuity between quasi-harmonic and relaxation oscillations, these extreme types of behavior actually occur in different technical applications and continued to be treated in separated contexts. Starting from relaxation oscillators and multivibrators (Abraham and Block, 1919) a particular branch of electronic circuit technology developed in the direction of such applications as time bases, counters, logical circuits, wave form generators, etc. On the other hand, the main concern in the design of harmonic oscillators was frequency stability; the way it is affected by nonlinearities was deduced from the principle of harmonic balance (Groszkowski, 1933) and the analysis of linear effects led to the development of the numerous oscillator circuits bearing the names of their inventors. All the above problems are, however, treated by elementary circuit analysis and are more related with device technology than with circuit theory.

Frequency conversion by amplitude modulation raised only elementary problems as long as tubes were used. With the introduction of rectifier modulators, such as the Cowan and ring modulators, in the early thirties, more difficult problems appeared, due to the interaction of all products whose amplitudes thus finally depend on the load impedances at all frequences. In carrier systems, modulators normally operate between highly selective filters, and a corresponding small signal theory for rectifier modulators between selective terminations was developed after 1939 by such workers as Caruthers, Kruse, Stieltjes, Tucker and Belevitch. The conception of a linear variable network as a linear network with an infinite number of ports (treating the impedance presented to each modulation product as a separate termination) did not yield tractable solutions in the case of nonselective terminations, and the case of RC loads (Belevitch, 1950) was treated by a direct method. The general problem became recently of major importance for the optimum design of filters for pulse-time modulated systems; after contributions by Cattermole (1958) and Desoer (1958), an analysis leading to a finite set of linear equations was obtained by Fettweis (1959). Another application of rectifier modulators was treated by Miller (1959) who showed that better frequency dividers could be designed by separating the various nonlinear effects which occur simultaneously in the tube of ordinary oscillators—in the circuits of Miller, the tube is a linear amplifier and the necessary nonlinear effects are produced by rectifier circuits separated by selective filters.

Harmonic generation by rectifier circuits does not seem to have been treated until quite recently—Page (1956) published a theorem on the maximum harmonic efficiency, and Belevitch and Neirynck (1957) described optimal circuits. On the contrary, harmonic production by non-linear reactance is not subject to the same restrictions, and magnetic harmonic generators (Peterson, Manley and Wrathall, 1937) are widely used. The theory of frequency conversion in magnetic modulators is more recent—power relations in nonlinear reactances were discussed by Manley and Rowe (1956), Page (1957), Duinker (1957) and many others. The possibility of amplification and the related negative resistance effects (Hartley) were known however, around 1920 and applied quite early to frequency dividers (transformation of a 60 cps supply into a 20 cps telephone ringing current) and to magnetic amplifiers. The recent revival of interest in parametric amplification is due to the availability of solid-state nonlinear capacitances. On the other hand, resonant circuits with nonlinear inductances, and the phenomenon of ferroresonance were also discussed before World War I, and led to Duffing's equation (1918). New interest in related phenomena was raised by the invention of the parametron (Goto, 1955).

The analysis of complicated nonlinear oscillator circuits by the classical methods of nonlinear mechanics is practically impossible and various approximate engineering methods have been developed, the first one being the so-called equivalent linearization of Krylov and Bogoliubov (1937). During the Second World War, a method based on the describing function has been introduced separately in the U.S.A., the U.S.S.R. and in France. In this method, one derives, from the instantaneous response of the nonlinear element to a harmonic excitation or to a combination of such excitations, an amplitude response or a set of amplitude responses, which are functions of the incident amplitudes; the linear part of the circuit is characterized by its transfer function or matrix. With the additional hypothesis that the linear circuits are highly selective, a finite set of algebraic equations is obtained for the steady-state amplitudes, and it is no longer necessary to consider explicitly any differential equations. This also holds true for the stability analysis of the steady-states, which is performed by linear perturbation methods, thus using only the standard criteria of linear network theory (Hurwitz, Nyquist). This "filter method" (as it is called in Russia) allowed a much simpler derivation of the classical results of van der Pol and others, and is now being successfully applied to new problems.