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RESEARCH ARTICLE

Finite-Time H_∞ Optimize Controller Design for Singular Positive Markovian Jump Delay Systems With Saturation Constraint

XUEJING REN [1], BO LI [2], AND JUNJIE ZHAO [2]
[1]School of Sciences, Changzhou Institute of Technology, Changzhou 213032, China
[2]School of Electrical and Information Engineering, Jiangsu University of Technology, Changzhou 213001, China

Corresponding author: Bo Li (libonanjing@aliyun.com)

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ABSTRACT In this article, the finite-time H_∞ control problem for singular positive Markovian jump systems with time-varying delay and saturation constraint was studied. Firstly, considering the discontinuities caused by the mode-dependent singular matrix and Markov jump switching behavior, a state feedback controller is designed to guarantee the positivity and mean-square locally finite-time H_∞ stability of the considered system. Secondly, the maximum finite-time domain of attraction of the considered system subject to input saturation is estimated. Finally, to show the effectiveness of our control strategy, the simulation results are given.

INDEX TERMS Finite-time H_∞ controller design, saturation constraint, singular positive systems, optimize control.

I. INTRODUCTION

In practice, many dynamic systems can be described by Markov jump systems, which often encounter random faults, component repairs and other factors, resulting from modifications of subsystem interconnection, abrupt environmental disturbance, and so on. In view of this, some important articles entirely devoted to many topics of this kind of system have been presented, including filter [1], [2], network [3], [4], actuator saturation [5], [6], [7], semi-Markovian jump systems [8], [9]. In the research results on T-S fuzzy mode [10], [11], [12], [13], fuzzy fault-tolerant tracking control of Markov jump systems with unknown mismatched faults was discussed in [12], and the mismatched quantized H_∞ output-feedback control of fuzzy Markov jump systems was studied in [13]. Considering the combination with event-based security control, event-based security control problem for an interconnected system with Markovian switching topologies was developed in [14], and event-triggered sliding mode secure control for nonlinear semi-Markov jump systems was established in [15]. In the last decades, a lot of scholars have been attracted to singular Markovian systems due to their extensive applications in the modeling of robotics, economics, and other areas [16], [17], [18], [19], [20]. It is worth noting that, the system state after and before the switching time may be discontinuous when considering the switching behavior and the singular matrix of the given system. However, few works are considered about such uncontinuity in [21], [22], and [23].

On the other hand, in some practical engineering applications, especially for some plants with short working time and fast response, the traditional lyapunov stability cannot achieve the desired control goal. In this background, finite-time control is becoming increasingly important. Furthermore, the finite-time control scheme for Markovian system has gradually become a hot research topic, and many meaningful results have been derived in [24], [25], [26], and [27]. Here are some references, sufficient conditions were obtained to guarantee that the singular T-S fuzzy Markovian jump system was finite time bounded in [26]. The finite-time stability problem of linear switched singular systems was addressed in [27], and new sufficient conditions

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were presented to guarantee the considered systems with finite-time unstable subsystems being finite-time bounded.

It should be noted that the physical system in the real world involves non negative variables, such as population level, absolute pressure, etc. These systems are represented as positive systems, which means that when the initial conditions and inputs are nonnegative, their state and output are nonnegative. Because of its practical application significance, the research of positive systems has received extensive attention [28], [29], [30]. For example, by applying an appropriate linear co-positive type Lyapunov-Krasovskii function, the state feedback controller was designed for singular positive Markovian jump systems in [30]. Furthermore, the problem of L_1 control for positive Markovian jump systems with partly known transition rates was discussed in [31]. The authors proposed an event-triggered control for positive Markov jump systems without/with input saturation in [32]. However, it is worth pointing out that how to stable singular positive Markovian jump systems with input saturation constraint is still a problem. Based on the above discussion, when considering the discontinuities caused by the singular matrix, the design of H_∞ finite-time controller for the singular positive Markovian jump system with time-varying delay and saturation constraint has not been fully investigated till now.

The main object of this article is to further investigate the finite-time H_∞ controller design method for singular positive Markovian jump delay systems with saturation constraint, and the key contributions of this paper are briefly summarized as follows:

i)When practical factors such as the discontinuities, partly unknown transition rates, time delay, disturbance signal and input saturation are combined in a Markovian jump system, while also considering the special characteristics of singular positive systems themselves, the controller designed method is given to guarantee that the considered system is positive and mean-square locally finite-time H_∞ stable.

ii)For input saturation, the maximum finite-time domain of attraction is estimated, and an optimization algorithm for solving the problem is proposed.

Notations:	
$A \geq 0$	The real symmetric and semi-positive definite matrix
$R^n(R^n_+)$	n-dimensional real(positive real) vector space
$(\Omega, \mathcal{F}, \mathcal{P})$	probability space
N^T	the transpose of the matrix N
$\lambda_{max/min}(A)$	the maximum/minimum element of matrix A

II. PROBLEM STATEMENTS AND PRELIMINARIES

In a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, consider singular Markovian jump systems (Σ):

$$E(r(t))\dot{x}(t) = A(r(t))x(t) + A_d(r(t))x(t-\zeta(t)) + B(r(t))sat(u(t)) + B_d(r(t))v(t), \tag{1}$$

$$y(t) = C_y(r(t))x(t) + C_{yd}(r(t))x(t-\zeta(t)) + D(r(t))v(t), \tag{2}$$

$$x(t) = \phi(t),\ t \in [-\bar{\zeta},\ 0], \tag{3}$$

in which $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^q$ represent the state vector and output vector, $u(t) \in \mathbb{R}^m$ represent the controlled input vector. $v(t)$ is the external input vector, $\phi(t) \in L_2^n[-\bar{\zeta}\ 0]$ is the vector-valued initial continuous function. $\zeta(t)$ is the unknown time-varying delay as follows:

$$0 < \zeta(t) < \bar{\zeta} < \infty,\ \dot{\zeta}(t) \leq \hat{\zeta} < 1, \tag{4}$$

$\{r(t)\}$ is a right continuous Markovian process and taking values in a finite set $S = \{1, 2, \ldots, N\}$. Transition probabilities is shown by:

$$P\{r(t+\Delta) = j\,|\,r(t) = i\} = \begin{cases} \lambda_{ij}\Delta + o(\Delta) & i \neq j, \\ 1 + \lambda_{ii}\Delta + o(\Delta) & i = j, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta\to 0}(o(\Delta)/\Delta) = 0$ and $\lambda_{ij} \geq 0$, for $j \neq i$, is the transition rate from mode i at time t to mode j at time $t+\Delta$ and

$$\lambda_{ii} = -\sum_{j\in S, j\neq i} \lambda_{ij}. \tag{5}$$

In this article, the transition rates may be partly unknown and we use "?" to denote the unknown part. For instance, the transition rates matrix can be described as follows:

$$P = \begin{bmatrix} \lambda_{11} & ? & \lambda_{13} & \cdots & \lambda_{1n} \\ \lambda_{21} & ? & \lambda_{23} & \cdots & ? \\ \vdots & \vdots & ? & \ddots & \vdots \\ \lambda_{n1} & & \lambda_{n3} & \ldots & ? \end{bmatrix}.$$

The set $S^i(\forall i \in S)$ is defined as:

$$S^i = S^i_k \bigcup S^i_{uk}$$

with $S^i_k \doteq \{j : \pi_{ij}\ \textit{is known for}\ j \in S\}$, and $S^i_{uk} \doteq \{j : \pi_{ij}\ \textit{is unknown for}\ j \in S\}$.

The saturation constraints are described as follows:

$$-u_{0(i)} \leq u_{(i)} \leq u_{0(i)},\ u_{0(i)} > 0,\ i = 1, \cdots, m. \tag{6}$$

To facilitate the presentation, we let $A_i = A(r(t))$, for each $r(t) = i \in S$, and other system constant matrices can be represented as A_{di}, B_i, B_{di}, C_i, C_{di}, E_i. It should be pointed out that E_i is known singular matrix. Then we design the controller as follows:

$$u(t) = K_{r(t)}x(t), \tag{7}$$

Taking Eq.(7) to Eq.(1), we have

$$E_i\dot{x}(t) = (A_i + B_iK_i)x(t) + A_{di}x(t-\zeta(t)) + B_i\vartheta(u(t)) + B_{di}v(t), \tag{8}$$

where $\vartheta(u(t)) = sat(u(t)) - u(t)$.

To complete the control objective, the following assumptions, definitions, and lemmas are needed.

Assumption 1: The external input vector $v(t)$ is bounded by $\hat{v}$ and satisfies:

$$\int_0^T \| v(t) \|_1 \, dt \le \hat{v}, \ \hat{v} \ge 0. \tag{9}$$

Definition 1 ([30]): For the initial condition $x_0 \ge 0$, $x_0(0-\zeta(t)) \ge 0$, if the corresponding trajectory $x(t) \ge 0$ holds for all $t > 0$, then systems (1)-(3) are said to be positive.

Lemma 1 ([21]): For every $i \in N$, the pair (E_i, A_i) is regularity and the absence of impulses if and only if there exist invertible matrices $\tilde{M}_i$ and $\tilde{N}_i$ such that

$$\tilde{M}_i E_i \tilde{N}_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{M}_i A_i \tilde{N}_i = \begin{bmatrix} \hat{A}_i & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Moreover, it can be found that matrices M_i and N_i satisfy the following equations,

$$\begin{aligned} \bar{E} &= M_i E_i N_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A}_i &= M_i A_i N_i = \begin{bmatrix} \bar{A}_{1i} & \bar{A}_{2i} \\ \bar{A}_{3i} & \bar{A}_{4i} \end{bmatrix}, \end{aligned} \tag{11}$$

then the pair (E_i, A_i) is impulse-free and regular if and only if $\bar{A}_{4i}$ is nonsingular and the above decomposition is satisfied.

Make $\bar{x}(t) = N_i^{-1}x(t)$, we can rewrite the system (8) as follows:

$$\begin{aligned} \bar{E}\dot{\bar{x}}(t) &= \tilde{A}_i\bar{x}(t) + \bar{A}_{di}\bar{x}(t-\zeta(t)) + \bar{B}_i\vartheta(u(t)) + \bar{B}_{di}v(t), \\ y(t) &= \bar{C}_{yi}\bar{x}(t) + \bar{C}_{ydi}\bar{x}(t-\zeta(t)) + D_i v(t), \end{aligned} \tag{12}$$

where $\bar{C}_{yi} = C_{yi}N_i, \ \bar{C}_{ydi} = C_{ydi}N_i, \ \bar{K}_i = K_i N_i$ and

$$\begin{aligned} \tilde{A}_i &= \bar{A}_i + \bar{B}_i\bar{K}_i = \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i2} \\ \tilde{A}_{i3} & \tilde{A}_{i4} \end{bmatrix}, \\ \bar{A}_{di} &= M_i A_{di} N_i = \begin{bmatrix} \bar{A}_{di1} & \bar{A}_{di2} \\ \bar{A}_{di3} & \bar{A}_{di4} \end{bmatrix}, \\ \bar{B}_i &= M_i B_i = \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \end{bmatrix}, \ \bar{B}_{di} = M_i B_{di} = \begin{bmatrix} \bar{B}_{di1} \\ \bar{B}_{di2} \end{bmatrix}. \end{aligned}$$

Due to switching behavior, the state of the system may be discontinuous before and after the switching jump time. Use $\bar{x}(t_j)^-$ and $\bar{x}(t_j)^+$ to represent the system states before and after switching moment t_j, respectively. If the considered system is regularity and the absence of impulses [17], it can be derived that:

$$\bar{x}(t_j)^+ = \Gamma_{ij}\bar{x}(t_j)^-, \tag{13}$$

with

$$\Gamma_{ij} = \begin{bmatrix} I & 0 \\ -(\tilde{A}_{j4})^{-1}\tilde{A}_{j3} & 0 \end{bmatrix} N_j^{-1}N_i.$$

Defining $\bar{x}^T = [\bar{x}_1{}^T, \ \bar{x}_2{}^T]$, the system (12) with $v(t) = 0$ can be reduced to the following nonsingular system:

$$\begin{aligned} \dot{\bar{x}}_1(t) &= \tilde{A}_{i1}\bar{x}_1(t) + \tilde{A}_{i2}\bar{x}_2(t) + \bar{A}_{di1}\bar{x}_1(t-\zeta(t)) \\ &\quad + \bar{A}_{di2}\bar{x}_2(t-\zeta(t)) + \bar{B}_{i1}\vartheta(u(t)) \\ 0 &= \tilde{A}_{i3}\bar{x}_1(t) + \tilde{A}_{i4}\bar{x}_2(t) + \bar{A}_{di3}\bar{x}_1(t-\zeta(t)) \\ &\quad + \bar{A}_{di4}\bar{x}_2(t-\zeta(t)) + \bar{B}_{i2}\vartheta(u(t)) \end{aligned} \tag{14}$$

Then, we have

$$\begin{aligned} \dot{\bar{x}}_1(t) &= (\tilde{A}_{i1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\tilde{A}_{i3})\bar{x}_1(t) \\ &\quad + (\bar{A}_{di1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{A}_{di3})\bar{x}_1(t-\zeta(t)) \\ &\quad + (\bar{A}_{di2} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{A}_{di4})\bar{x}_2(t-\zeta(t)) \\ &\quad + (\bar{B}_{i1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{B}_{i2})\vartheta(u(t)) \\ \bar{x}_2(t) &= -\tilde{A}_{i4}^{-1}\tilde{A}_{i3}\bar{x}_1(t) - \tilde{A}_{i4}^{-1}\bar{A}_{di3}\bar{x}_1(t-\zeta(t)) \\ &\quad - \tilde{A}_{i4}^{-1}\bar{A}_{di4}\bar{x}_2(t-\zeta(t)) - \tilde{A}_{i4}^{-1}\bar{B}_{i2}\vartheta(u(t)) \end{aligned} \tag{15}$$

Obviously, the system (15) is positive if and only if $(\tilde{A}_{i1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\tilde{A}_{i3})$ are Metzler matrices and

$$\begin{aligned} &\bar{A}_{di1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{A}_{di3} \ge 0, \ \bar{A}_{di2} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{A}_{di4} \ge 0, \\ &\quad -\tilde{A}_{i4}^{-1}\tilde{A}_{i3} \ge 0, \ -\tilde{A}_{i4}^{-1}\bar{A}_{di3} \ge 0, \ -\tilde{A}_{i4}^{-1}\bar{A}_{di4} \ge 0, \\ &\quad -\tilde{A}_{i4}^{-1}\bar{B}_{i2} \ge 0, \ \text{if } \vartheta(u(t)) \ge 0, \\ &\quad -\tilde{A}_{i4}^{-1}\bar{B}_{i2} \le 0, \ \text{if } \vartheta(u(t)) \le 0, \\ &(\bar{B}_{i1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{B}_{i2}) \ge 0, \ \text{if } \vartheta(u(t)) \ge 0, \\ &(\bar{B}_{i1} - \tilde{A}_{i2}\tilde{A}_{i4}^{-1}\bar{B}_{i2}) \le 0, \ \text{if } \vartheta(u(t)) \le 0. \end{aligned} \tag{16}$$

Definition 2 ([30]): Regularity and the absence of impulses.

(i) System (12) with v(t) = 0 is said to be regularity, if $det(s\bar{E} - \bar{A}_i) \neq 0$ for all $t \in [0, \ T]$.

(ii) System (12) with v(t) = 0 is said to be the absence of impulses, if $deg(det(s\bar{E} - \bar{A}_i)) = rank(\bar{E})$ for all $t \in [0, \ T]$.

Lemma 2 ([21]): If the system (12) is regularity and the absence of impulses, then the following matrices transformation can be fulfilled for nonsingular matrix $\bar{M}_i$ and $\bar{N}_i$,

$$\begin{aligned} &\bar{M}_i\bar{E}\bar{N}_i = \bar{E} = diag\{I, 0\}, \ \bar{M}_i\tilde{A}_i\bar{N}_i = diag\{\Xi_{Ai}, I\}, \\ &\bar{M}_i\bar{A}_{di}\bar{N}_i = diag\{\Xi_{Adi}, 0\}, \\ &\bar{M}_i\bar{B}_i = \begin{bmatrix} \Xi_{Bi} \\ 0 \end{bmatrix}, \ \bar{M}_i\bar{B}_{di} = \begin{bmatrix} \Xi_{Bdi} \\ 0 \end{bmatrix}, \end{aligned}$$

and the system (15) can be equivalent to the following system

$$\begin{aligned} \dot{\bar{x}}(t) &= \tilde{\Xi}_{Ai}\bar{x}(t) + \tilde{\Xi}_{Adi}\bar{x}(t-\zeta(t)) + \tilde{\Xi}_{Bi}\vartheta(u(t)) \\ &\quad + \tilde{\Xi}_{Bdi}v(t), \\ \bar{x}(t_j)^+ &= \Gamma_{ij}\bar{x}(t_j)^-, \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\tilde{\Xi}_{Ai} = \bar{N}_i\begin{bmatrix} \Xi_{Ai} & 0 \\ 0 & 0 \end{bmatrix}\bar{N}_i^{-1}, \ \tilde{\Xi}_{Bdi} = \bar{N}_i\begin{bmatrix} \Xi_{Bdi} \\ 0 \end{bmatrix}, \\ &\tilde{\Xi}_{Bi} = \bar{N}_i\begin{bmatrix} \Xi_{Bi} \\ 0 \end{bmatrix}, \ \tilde{\Xi}_{Adi} = \bar{N}_i\begin{bmatrix} \Xi_{Adi} & 0 \\ 0 & 0 \end{bmatrix}\bar{N}_i^{-1}. \end{aligned}$$

Lemma 3: For the system (12) and the designed controller parameter $\bar{K}_i$, the given appropriate matrix $L_i \in \mathbb{R}^{m\times n}$, if $\bar{x}(t)$ is in the set $D(u_o)$ which is defined as follows:

$$\begin{aligned} D(u_o) = \{&\bar{x}(t) \in \mathbb{R}^n; \ -u_{0(k)} \le (\bar{K}_{i(k)} + L_{i(k)})\bar{x}(t) \le u_{0(k)}, \\ &u_{0(k)} > 0, \ k = 1, \dots, m\}, \end{aligned}$$

then for any positive matrix $T_i \in \mathbb{R}^n$, we derive:

$$\vartheta(u(t))^T T_i - \bar{x}(t)^T L_i^T T_i$$

$$\geq 0,\ if\ \vartheta(u(t)) \leq 0,$$
$$-\vartheta(u(t))^T T_i + \bar{x}(t)^T L_i^T T_i \geq 0,\ if\ \vartheta(u(t)) \geq 0.$$

Proof: As can be seen from the above, $u(t) = \bar{K}_i\bar{x}(t) = sat(u(t)) - \vartheta(u(t))$ holds, then one has

$$\begin{aligned}-u_{0(k)} &\leq (sat(u(t)) - \vartheta(u(t))) + L_{i(k)}\bar{x}(t) \leq u_{0(k)} \\ &\Rightarrow -u_{0(k)} - sat(u(t)) \leq -\vartheta(u(t)) + L_{i(k)}\bar{x}(t) \\ &\leq u_{0(k)} - sat(u(t)).\end{aligned} \tag{18}$$

Consider that, when saturation occurs, we have $u_{0(k)} - \mid sat(u(t)) \mid = 0$, if $\vartheta(u(t)) \leq 0(u(t) \geq 0)$, one can derive

$$\begin{aligned}&-\vartheta(u(t)) + L_{i(k)}\bar{x}(t) \leq u_{0(k)} - sat(u(t)) \\ &\Rightarrow \vartheta(u(t)) - L_{i(k)}\bar{x}(t) \geq 0 \\ &\Rightarrow \vartheta(u(t))^T T_i - \bar{x}(t)^T L_i^T T_i \geq 0,\end{aligned} \tag{19}$$

if $\vartheta(u(t)) \geq 0(u(t) \leq 0)$, we have

$$\begin{aligned}&-\vartheta(u(t)) + L_{i(k)}\bar{x}(t) \geq 0 \\ &\Rightarrow -\vartheta(u(t))^T T_i + \bar{x}(t)^T L_i^T T_i \geq 0.\end{aligned} \tag{20}$$

The proof is completed.

Definition 3: If there exist constant $\alpha > 0$ and $\beta > 0$, such that $\varepsilon(\|\bar{E}\bar{x}(t;\bar{x_0}, r_0)\|_1) \leq \alpha e^{-\beta t}\varepsilon(\|\bar{E}\bar{x_0}\|_1)$, the system (12) is exponentially stable in the mean square sense.

Definition 4 ([27]): For system specified parameters $c_2 > c_1 > 0$, $T > 0$ and mode-dependent matrix $\hat{R}_i > 0$, if a controller with the same form as formula(7) and the state trajectory of the system satisfies:

$$\varepsilon\{\bar{x}^T(t_1)E^T\hat{R}_i\} \leq c_1 \Rightarrow \varepsilon\{\bar{x}^T(t_2)E^T\hat{R}_i\} \leq c_2,$$
$$t_1 \in [-\zeta,\ 0],\ t_2 \in [0,\ T], \tag{21}$$

the system (12) is stochastically finite-time bounded stable under $v(t) \neq 0$ and $(c_1,\ c_2,\ T,\ \hat{R}_i,\ \hat{v})$ conditions.

Definition 5 ([20]): For system (12) and any system mode i, construct the stochastic Lyapunov-Krasovskii function $V(\bar{x}(t),\ r(t),\ t > 0)$, and along the system (12) its weak infinitesimal operator is represented as:

$$\begin{aligned}&LV(\bar{x}(t),\ i,\ t) \\ &= lim_{\Delta t\to 0}\frac{1}{\Delta t}[\varepsilon\{V(\bar{x}(t+\Delta t),\ r_{t+\Delta t},\ t+\Delta t)|\bar{x}(t) = x, \\ &\quad r_t = i\} - V(\bar{x}(t),\ i,\ t)] \\ &= \frac{\partial}{\partial t}V(\bar{x}(t),\ i,\ t) + \frac{\partial}{\partial x}V(\bar{x}(t),\ i,\ t)\dot{x}(t,\ i) \\ &\quad + \sum_{j=1}^{N}\pi_{ij}V(\bar{x}(t),\ j,\ t).\end{aligned} \tag{22}$$

III. MAIN RESULTS

In this part, we devote to the exponential stability and finite-time H_∞ performance analysis for systems (12).

Theorem 1: For a given scalar $\lambda > 0$, given matrix L_i, the systems(12) with $v(t) = 0$ and suitable initial conditions belonging to $\varepsilon(\bar{E}^T P_i,\ 1)$ is positive and exponentially stable, if there exists matrix $P_i \in R_+^n$, $T_i \in R_+^n$, such that

(i) The pair $(\bar{E},\ \tilde{A}_i)$ is regularity and the absence of impulses;

(ii) Condition (16) and the following inequality hold

$$\begin{aligned}&\Sigma_{j=1,j\neq i}^N\lambda_{ij}\Upsilon_{ij}^T\bar{E}^T P_j + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}_i^T P_i + \bar{E}Q \\ &\quad + L_i^T T_i < 0,\ if\ \vartheta(u(t)) \geq 0,\end{aligned} \tag{23}$$

$$\begin{aligned}&\Sigma_{j=1,j\neq i}^N\lambda_{ij}\Upsilon_{ij}^T\bar{E}^T P_j + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}_i^T P_i + \bar{E}Q \\ &\quad - L_i^T T_i < 0,\ if\ \vartheta(u(t)) \leq 0,\end{aligned} \tag{24}$$

$$\bar{A}_{di}^T P_i - (1-\hat{\zeta})\bar{E}Q < 0, \tag{25}$$

$$-T_i + \bar{B}_i^T P_i < 0,\ if\ \vartheta(u(t)) \geq 0, \tag{26}$$

$$-T_i - \bar{B}_i^T P_i < 0,\ if\ \vartheta(u(t)) \leq 0, \tag{27}$$

$$u_{0(k)}\bar{E}^T P_i - (\bar{K}_{i(k)}^T + L_{i(k)}^T) \geq 0,\ if\ \vartheta(u(t)) \leq 0, \tag{28}$$

$$u_{0(k)}\bar{E}^T P_i + (\bar{K}_{i(k)}^T + L_{i(k)}^T) \geq 0,\ if\ \vartheta(u(t)) \geq 0. \tag{29}$$

Proof: For any system mode $r(t) = i \in S$, the following Lyapuonv function is considered for system(12):

$$V(\bar{x}(t), r(t)) = \bar{x}(t)^T\bar{E}^T P_{r(t)} + \int_{t-\zeta(t)}^{t}\bar{x}(s)^T\bar{E}Qds,$$

set $r(t) = i$, and $P_i \in R_+^n$, $Q \in R_+^n$, one can drive that

$$\begin{aligned}&\varepsilon(V(\bar{x}(t+\Delta t), r(t+\Delta t)|\bar{x}(t), r(t)) - V(\bar{x}(t), r(t))) \\ &= \Sigma_{j=1,j\neq i}^N\lambda_{ij}\Delta t\bar{x}(t+\Delta t)^T\bar{E}^T P_j \\ &\quad + (1+\lambda_{ii}\Delta t)\bar{x}(t+\Delta t)^T\bar{E}^T P_i - \bar{x}(t)^T\bar{E}^T P_i \\ &\quad + (\bar{x}(t)^T\bar{E}Q - (1-\dot{\zeta}(t))\bar{x}(t-\zeta(t))^T\bar{E}Q)\Delta t\end{aligned} \tag{30}$$

When $i = j$, applying $\bar{E}\bar{x}(t+\Delta t) = \bar{E}\bar{x}(t) + (\tilde{A}_i\bar{x}(t) + \bar{A}_{di}\bar{x}(t-\zeta(t)) + \bar{B}_i\vartheta(u(t)))\Delta t$, one can get from Eq.(30) that

$$\begin{aligned}&(1+\lambda_{ii}\Delta t)\bar{x}(t+\Delta t)^T\bar{E}^T P_i - \bar{x}(t)^T\bar{E}^T P_i \\ &= (1+\lambda_{ii}\Delta t)(\bar{x}^T(t)\bar{E}^T + (\bar{x}^T(t)\tilde{A}_i^T + \bar{x}^T(t-\zeta(t))\bar{A}_{di}^T \\ &\quad + \vartheta(u(t))^T\bar{B}_i^T)P_i\Delta t - \bar{x}(t)^T\bar{E}^T P_i \\ &= \lambda_{ii}\Delta t\bar{x}(t)^T\bar{E}^T P_i + (\bar{x}^T(t)\tilde{A}_i^T + \bar{x}^T(t-\zeta(t))\bar{A}_{di}^T \\ &\quad + \vartheta(u(t))^T\bar{B}_i^T)P_i\Delta t + o(\Delta t).\end{aligned} \tag{31}$$

When $j \neq i$, we should deal with the discontinuities. From Eq.(17), we have $\bar{x}(t+\Delta t) = \Gamma_{ij}(\bar{x}(t) + (\tilde{\Xi}_{Ai}x(t) + \tilde{\Xi}_{Adi}x(t-\zeta(t)) + \tilde{\Xi}_{Bi}\vartheta(u(t)))\Delta t)$, then one can deduce

$$\begin{aligned}&\Sigma_{j=1,j\neq i}^N\lambda_{ij}\Delta t\bar{x}(t+\Delta t)^T\bar{E}^T P_j \\ &= \Sigma_{j=1,j\neq i}^N\lambda_{ij}\Delta t\bar{x}(t)^T(\Gamma_{ij})^T\bar{E}^T P_j + o(\Delta t)\end{aligned} \tag{32}$$

Apply $\dot{\zeta}(t) < \hat{\zeta}$ and Lemma 3, we can further obtain

$$\begin{aligned}&\varepsilon(V(\bar{x}(t+\Delta t), r(t+\Delta t, \sigma(t+\Delta t)|\bar{x}(t), r(t), \sigma(t)) \\ &\quad - V(\bar{x}(t), r(t), \sigma(t))), \\ &\leq (31) + (32) + (\bar{x}(t)^T\bar{E}Q - (1-\hat{\zeta})\bar{x}(t-\zeta(t))^T\bar{E}Q)\Delta t \\ &\quad + (\vartheta(u(t)))^T T_i - \bar{x}(t)^T L_i^T T_i)\Delta t,\ ,\ if\ \vartheta(u(t)) \leq 0, \\ &\quad - (\vartheta(u(t)))^T T_i - \bar{x}(t)^T L_i^T T_i)\Delta t,\ if\ \vartheta(u(t)) \geq 0.\end{aligned} \tag{33}$$

Denoting

$$\Upsilon_{ij} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} N_j^{-1}N_i,$$

it's thus clear that $\bar{E}^T\Gamma_{ij}^q = \bar{E}^T\Upsilon_{ij}$. From conditions (23)-(27) of Theorem 1, we have

$$LV(\bar{x}(t), r(t)) < \bar{x}(t)^T\bar{E}^T P_i \leq 0. \tag{34}$$

Denote $U_i = \Sigma_{j=1,j\neq i}^N \lambda_{ij}\Upsilon_{ij}^T\bar{E}^T P_j + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}_i^T P_i + \bar{E}Q \pm L_i^T T_i$, and $\delta_1 = \lambda_{min}(-U_i)$, δ_2 is the minimum value of all non-zero elements of $(\bar{E}^T P_i)$, $\delta_3 = \lambda_{max}(\bar{E}^T P_i)$, $\delta_4 = \lambda_{max}(Q)$, $\| \bar{x}(0-\zeta(t)) \|_1 \leq \| \bar{x}(0) \|_1$. One can deduce that

$$LV(\bar{x}(t), r(t)) \leq -\delta_1\|\bar{x}(t)\|_1. \tag{35}$$

Meanwhile, it's known that

$$\varepsilon V(\bar{x}(t), r(t)) \geq \delta_2\varepsilon\|\bar{E}\bar{x}(t)\|_1. \tag{36}$$

One can further get from Dynkin Lemma

$$\varepsilon V(\bar{x}(t), r(t)) \leq \varepsilon V(\bar{x}(0), r(0)) - \delta_1\varepsilon(\int_0^t \|\bar{x}(s)\|_1 ds). \tag{37}$$

By combining the above inequalities, it can be concluded that

$$\begin{aligned}
\delta_2\varepsilon\|\bar{E}\bar{x}(t)\|_1 &\leq \varepsilon\delta_3\|\bar{E}\bar{x}(0)\|_1 + \varepsilon\delta_4\bar{\zeta}\|\bar{E}\bar{x}(0)\|_1 \\
&\quad - \delta_1\varepsilon(\int_0^t \|\bar{x}(s)\|_1 ds) \\
\varepsilon\|\bar{E}\bar{x}(t)\|_1 &\leq \delta_2^{-1}(\delta_3+\delta_4\bar{\zeta})\varepsilon\|\bar{E}\bar{x}(0)\|_1 \\
&\quad - \delta_2^{-1}\delta_1\varepsilon(\int_0^t \|\bar{E}\bar{x}(s)\|_1 ds) \\
\varepsilon\|\bar{E}\bar{x}(t)\|_1 &\leq \delta_2^{-1}e^{-\delta_1(\delta_3+\delta_4\bar{\zeta})^{-1}}\varepsilon\|\bar{E}\bar{x}(0)\|_1,
\end{aligned} \tag{38}$$

then the system (12) is exponentially stable in the mean square sense based on Definition 3. Define the initial state $\bar{x}(t) \in \varepsilon(\bar{E}^T P_i,\ 1)$, which means that $\bar{x}^T(t)\bar{E}^T P_i\bar{x}(t) \leq 1$. From conditions (28)-(29), with $\vartheta(u(t)) \leq 0$ it's easily known that

$$\begin{aligned}
&u_{0(k)}\bar{x}^T(t)\bar{E}^T P_i - \bar{x}^T(t)(\bar{K}_{i(k)}^T + L_{i(k)}^T) \geq 0 \\
&\Rightarrow u_{0(k)} \geq (\bar{x}^T(t)\bar{E}^T P_i)^{-1}\bar{x}^T(t)(\bar{K}_{i(k)}^T + L_{i(k)}^T) \\
&\Rightarrow u_{0(k)} \geq \bar{x}^T(t)(\bar{K}_{i(k)}^T + L_{i(k)}^T).
\end{aligned} \tag{39}$$

If $\vartheta(u(t)) \geq 0$, we have

$$\begin{aligned}
&u_{0(k)}\bar{x}^T(t)\bar{E}^T P_i + \bar{x}^T(t)(\bar{K}_{i(k)}^T + L_{i(k)}^T) \geq 0 \\
&\Rightarrow \bar{x}^T(t)(\bar{K}_{i(k)}^T + L_{i(k)}^T) \geq -u_{0(k)}\bar{x}^T(t)\bar{E}^T P_i \\
&\Rightarrow \bar{x}^T(t)(\bar{K}_{i(k)}^T + L_{i(k)}^T) \geq -u_{0(k)}.
\end{aligned} \tag{40}$$

It is known that $\varepsilon(\bar{E}^T P_i,\ 1) \in D(u(0))$ which defined in Lemma 3.

Theorem 2: For some known parameters, $\lambda > 0$, $\alpha > 0$, if there exists matrix $P_i \in R_+^n$, $T_i \in R_+^n$, and $S \in R_+^n$ such that

(i) The pair $(\bar{E},\ \tilde{A}_i)$ is regularity and the absence of impulses;

(ii) Condition (16), (25)-(29) and the following inequality hold

$$\begin{aligned}
&\Sigma_{j=1,j\neq i}^N \lambda_{ij}\Upsilon_{ij}^T\bar{E}^T P_j + \alpha S + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}_i^T P_i \\
&\quad + \bar{E}Q + L_i^T T_i < 0, \text{if } \vartheta(u(t)) \geq 0,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\Sigma_{j=1,j\neq i}^N \lambda_{ij}\Upsilon_{ij}^T\bar{E}^T P_j + \alpha S + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}_i^T P_i \\
&\quad + \bar{E}Q - L_i^T T_i < 0, \text{if } \vartheta(u(t)) \leq 0,
\end{aligned} \tag{42}$$

$$\bar{B}_{di}^T P_i - \alpha S < 0, \tag{43}$$

$$c_1\sigma_P + c_1\zeta\sigma_Q + \alpha\hat{v}\sigma_S(1-e^{-aT}) < e^{-aT}\sigma_p c_2, \tag{44}$$

where $\sigma_P = max_{i\in S}\sigma_{max}(\bar{P}_i)$, $\sigma_p = min_{i\in S}\sigma_{min}(\bar{P}_i)$, $\sigma_Q = \sigma_{max}(\bar{Q})$, $\sigma_q = \sigma_{min}(\bar{Q})$, $\sigma_S = \sigma_{max}(S)$, $\bar{Q} = \bar{R}_i Q$, $\bar{P}_i = \bar{R}_i P_i$, the systems(12) with $v(t) \neq 0$ is locally finite-time stochastically stable with $(c_1\ c_2\ T\ R_i\ \hat{v})$, and incipient states within the scope of $\varepsilon(\bar{E}^T P_i,\ 1)$, where

$$R_i = \begin{bmatrix} R_{i1} \\ R_{i2} \\ \vdots \\ R_{in} \end{bmatrix},\ \bar{R}_i = \begin{bmatrix} \frac{1}{R_{i1}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{R_{i2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{R_{in}} \end{bmatrix}.$$

Proof: Using Lyapunov functionals similar to those in Theorem 1, and combining the conditions in theorem 2, one can obtain

$$LV(\bar{x}(t),\ i) < \alpha v^T(t)S + \alpha(V(\bar{x}(t),\ i) - \int_{t-\zeta}^t \bar{x}(s)^T\bar{E}Qds. \tag{45}$$

Since that $-\int_{t-\zeta}^t \bar{x}(s)^T\bar{E}Qds \leq 0$, the following formula is shown as

$$LV(\bar{x}(t),\ i) < \alpha v^T(t)S + \alpha V(\bar{x}(t),\ i). \tag{46}$$

Multiplying left by $e^{-\alpha t}$ yields,

$$L[e^{-\alpha t}V(\bar{x}(t),\ i)] < \alpha e^{-\alpha t}v^T(t)S. \tag{47}$$

Integral calculation from 0 to t,

$$e^{-\alpha t}V(\bar{x}(t),\ i) - V(\bar{x}(0),\ r_0) < \alpha\int_0^t e^{-\alpha s}v^T(t)ds. \tag{48}$$

So it can be concluded that

$$\begin{aligned}
\varepsilon\{\bar{x}(t)^T\bar{E}^T P_i\} &\leq V(\bar{x}(t),\ i) \\
&< e^{\alpha t}V(\bar{x}(0),\ r_0) + \alpha\hat{v}\sigma_S e^{\alpha t}\int_0^t e^{-\alpha s}ds \\
&< e^{\alpha t}[\alpha\hat{v}\sigma_S(1-e^{\alpha t}) + V(\bar{x}(0),\ r_0)]
\end{aligned} \tag{49}$$

Meanwhile, it can be inferred

$$\begin{aligned}
\varepsilon\{\bar{x}(t)^T\bar{E}^T P_i\} &= \varepsilon\{\bar{x}(t)^T\bar{E}^T\bar{R}_i^{-1}\bar{R}_i P_i\} \\
&= \varepsilon\{\bar{x}(t)^T\bar{E}^T\bar{R}_i^{-1}\bar{P}_i\}
\end{aligned}$$

$$\Rightarrow \varepsilon\{\bar{x}(t)^T\bar{E}^T P_i\} \geq \varepsilon\{\sigma_p\bar{x}(t)^T\bar{E}^T R_i\} \tag{50}$$

$$\Rightarrow \varepsilon\{\bar{x}(t)^T\bar{E}^T P_i\} \leq \varepsilon\{\sigma_P\bar{x}(t)^T\bar{E}^T R_i\} \tag{51}$$

Further calculation reveals that

$$\varepsilon\{\sigma_q\bar{x}(t)^T R_i\} \leq \varepsilon\{\bar{x}(t)^T\bar{Q}\} \leq \varepsilon\{\sigma_Q\bar{x}(t)^T R_i\} \tag{52}$$

Because of $\varepsilon\{\bar{x}^T(0)\bar{E}^T R_i\} \leq c_1$,

$$\varepsilon\{\bar{x}(t)^T\bar{E}^T P_i\} < e^{\alpha t}[c_1\sigma_P + c_1\zeta\sigma_Q + \alpha\hat{v}\sigma_s(1-e^{\alpha t})]. \tag{53}$$

Being able to know that

$$\varepsilon\{\bar{x}(t)^T\bar{E}^T P_i\} \geq \sigma_p\varepsilon\{\bar{x}(t)^T\bar{E}^T R_i\},$$

one can achieved that

$$\varepsilon\{\bar{x}(t)^T\bar{E}^T R_i\} < \frac{e^{\alpha t}[c_1\sigma_P + c_1\zeta\sigma_Q + \alpha\hat{v}\sigma_s(1-e^{\alpha t})]}{\sigma_p}. \tag{54}$$

Conditions (44) shows that $\varepsilon\{\bar{x}(t)^T\bar{E}^T R_i\} < c_2$.

Similar to Theorem 1, the theorem is proven.

The following conclusion, we further consider the case where the transition rates are partially unknown.

Theorem 3: For some known parameters, $\lambda_1 > 0$, $\lambda > 0$, $\alpha > 0$, if there exists matrix $P_i \in R^n_+$, $\rho_i \in R^n_+$, $q_i \in R^q_+$, $T_i \in R^n_+$, and $S \in R^n_+$ such that

(i) The pair $(\bar{E},\ \tilde{A}_i)$ is regularity and the absence of impulses;

(ii) Condition (16), (26)-(27) and the following inequality hold

$$U_{i1} < 0,\ if\ \vartheta(u(t)) \geq 0, \tag{55}$$
$$U_{i2} < 0,\ if\ \vartheta(u(t)) \leq 0, \tag{56}$$
$$\bar{E}^T P_i - \bar{E}^T\rho_i \leq 0,\ i \neq j \in S_{uk}, \tag{57}$$
$$\bar{E}^T P_i - \bar{E}^T\rho_i \geq 0,\ i = j \in S_{uk}, \tag{58}$$
$$\bar{A}^T_{di}P_i + \bar{C}^T_{di}q_i - (1-\hat{\zeta})\bar{E}Q < 0, \tag{59}$$
$$(D_i^T + \bar{B}^T_{di})P_i - \gamma I < 0, \tag{60}$$
$$\frac{1}{\lambda_1}u_{0(k)}\bar{E}^T P_i - (\bar{K}^T_{i(k)} + L^T_{i(k)}) \geq 0,\ if\ \vartheta(u(t)) \leq 0, \tag{61}$$
$$\frac{1}{\lambda_1}u_{0(k)}\bar{E}^T P_i + (\bar{K}^T_{i(k)} + L^T_{i(k)}) \geq 0,\ if\ \vartheta(u(t)) \geq 0, \tag{62}$$
$$c_1\sigma_P + c_1\zeta\sigma_Q + \alpha\gamma\hat{v}(1-e^{-aT}) < e^{-aT}\sigma_p c_2, \tag{63}$$

where

$$\begin{aligned} U_{i1} &= \Sigma^N_{j=1,j\neq i}\lambda_{ij}\Upsilon^T_{ij}\bar{E}^T(P_j-\rho_i) + \alpha S + \bar{C}^T_i q_i \\ &\quad + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}^T_i P_i + \bar{E}Q + L^T_i T_i, \\ U_{i2} &= \Sigma^N_{j=1,j\neq i}\lambda_{ij}\Upsilon^T_{ij}\bar{E}^T(P_j-\rho_i) + \alpha S + \bar{C}^T_i q_i \\ &\quad + (\lambda+\lambda_{ii})\bar{E}^T P_i + \tilde{A}^T_i P_i + \bar{E}Q - L^T_i T_i, \end{aligned}$$

the systems(12) with $v(t) \neq 0$ and initial conditions of system satisfied with $\varepsilon(\bar{E}^T P_i,\ \lambda_1)$ is positive and locally finite-time stochastically stable with $(c_1\ c_2\ T\ R_i\ \hat{v})$, and incipient states within the scope of $\varepsilon(\bar{E}^T P_i,\ 1)$.

Proof: Based on the transition rates of the Markov jump process partially unknown and $\Sigma^N_{j=1}\lambda_{ij} = 0$, choose a Lyapuonv-Krasovkii function similar to Theorem 1 and split the transition rates into known and unknown parts, then we have

$$\begin{aligned} LV(\bar{x}(t),\ i) &= LV(\bar{x}(t),\ i) - \Sigma^N_{j=1}\lambda_{ij}\bar{E}^T\rho_i = LV(\bar{x}(t),\ i) \\ &\quad - \Sigma^N_{j=1,j\in S_k}\lambda_{ij}\bar{E}^T\rho_i - \Sigma^N_{j=1,j\in S_{uk}}\lambda_{ij}\bar{E}^T\rho_i \end{aligned} \tag{64}$$

From theorem 2, it can be derived that,

$$LV(\bar{x}(t),\ i) < \gamma\|v(t)\|_1 - y^T(t)q_i + \alpha V(\bar{x}((t),\ i). \tag{65}$$

On account of $-y(t)^T q_i < 0$, we conclude

$$LV(\bar{x}(t),\ i) < \gamma\|v(t)\|_1 + \alpha V(\bar{x}((t),\ i). \tag{66}$$

Similar to the proof of Theorem 2, Equation (63) yields. At the same time, the following equation holds under zero initial conditions,

$$e^{-\alpha t}V(\bar{x}(t),\ i) < \int_0^T(\gamma\|v(t)\|_1 - y^T(t)q_i)dt. \tag{67}$$

Further derivation reveals that

$$\int_0^T y(t)^T q_i dt \leq \int_0^T \gamma\|v(t)\|_1 dt, \tag{68}$$

and

$$\int_0^T \|y(t)\|_1 dt \leq \frac{\gamma}{\lambda_{min}(q_i)}\int_0^T \|v(t)\|_1 dt. \tag{69}$$

Given that the initial state $\bar{x}(t)$ belongs to $\varepsilon(\bar{E}^T P_i,\ \lambda_1)$ and $\bar{x}^T(t)\bar{E}^T P_i \leq \lambda_1$. it can be concluded that $\varepsilon(\bar{E}^T P_i,\ \lambda_1) \in D(u(0))$ which is mentioned in Lemma 3. The theorem has been proven so far.

Remark 1: The gains of the designed controller can be figured out as $\tilde{K}_i = \bar{K}^T_i\bar{B}^T_i P_i$ by calculating conditions (55)-(63). The following are the specific calculation steps:

Firstly, select the λ value, and then use the LMI toolbox to find unknown matrices that satisfy the linear matrix inequality (55)-(63), such as $\tilde{K}_i$, P_i;

Secondly, use the $\tilde{K}_i = \bar{K}^T_i\bar{B}^T_i P_i$ obtained in the previous step to test whether the (i) and (ii) conditions in Theorem 3 are true. If it holds, then the feedback gain matrices $\bar{K}_i$ are the feasible solution; On the contrary, adjust the value of λ and return to the first step.

Remark 2: The method of optimizing the finite time domain of attraction can be expressed as:

$$\begin{aligned} &\min \quad \nu \\ &\quad (\bar{K}_i,\ c_1,\ c_2,\ \nu) \\ &s.t.\ inequalities\ (55)-(63)\ with\ \nu = \frac{1}{\lambda_1}. \end{aligned}$$

IV. NUMERICAL EXAMPLES

Example 1: The given system(Σ) (1)-(3) parameters as the follows:

Mode 1

$$A_1 = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.8 & 1.1 \\ 0 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$B_{d1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_1 = \begin{bmatrix} -1 & -1 \end{bmatrix},\ C_{d1} = \begin{bmatrix} 1 & -1 \end{bmatrix},$$
$$D_1 = 1,\ E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Mode 2

$$A_2 = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.8 & 0.3 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$B_{d2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix},\ C_{d2} = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

$$D_2 = 1,\ E_2 = \begin{bmatrix} 1 & 0 \\ 0 & q0 \end{bmatrix}.$$

Choose

$$M_2 = M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ N_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$N_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Then one can be obtained

$$\bar{E} = M_1 E_1 N_1 = M_2 E_2 N_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\bar{A}_1 = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix},\ \bar{A}_2 = \begin{bmatrix} 3 & 1 \\ -3 & 1 \end{bmatrix},$$
$$\bar{A}_{d1} = \begin{bmatrix} 0.8 & 0.3 \\ 0 & 0 \end{bmatrix},\ \bar{A}_{d2} = \begin{bmatrix} 0.3 & 0.6 \\ 0 & 0 \end{bmatrix}.$$

In this system simulation, the initial values of the system are set to external disturbances as $v(t) = 0$, input restriction as $|u_t| \leq 0.5$, and the delay as $\zeta(t) = 0.5|sin\ t|$. Because the range of values for sine functions is between 0 and 1, it's easy to know $\bar{\zeta} = \hat{\zeta} = 0.5$.

In addition, the following values are known in advance as follows:

$$\phi(t) = \begin{bmatrix} 0.5t + 0.5 \\ 2t + 1.5 \end{bmatrix}, \quad \lambda_{ij} = \begin{bmatrix} -0.6 & 0.6 \\ 0.3 & -0.3 \end{bmatrix}.$$

By using theorem 1, we have

$$K_1 = \begin{bmatrix} -5.3421 & -1.8872 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -4.8763 & -0.9785 \end{bmatrix}.$$

In simulation, time delay and input limitation can represent communication delay and actuator saturation limitation in actual systems, respectively. After comparing *Fig*. 1 and *Fig*. 2, it can be concluded that the controller designed by the algorithm in the article can make the closed-loop system exponentially stable and the system state converges to 0.

Example 2: Using the same system model parameters and input limitation given in Example 1, the designed metrics with finite-time stability are given as $c_1 = 0.4,\ c_2 = 0.8,\ T = 10,\ R_i = I_1,\ \gamma = 0.8,\ v(t) \leq \hat{v} = 1$. Given initial system values and partially unknown system transition rate matrix as follows:

$$\phi(t) = \begin{bmatrix} -0.5t \\ -t \end{bmatrix}. \quad \lambda_{ij} = \begin{bmatrix} ? & ? \\ 0.4 & -0.4 \end{bmatrix}.$$

According to the linear matrix inequality optimization result of Theorem 3, it can be obtained that

$$K_1 = \begin{bmatrix} -7.4736 & -1.2561 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -6.3826 & -1.0024 \end{bmatrix}.$$

Similar to *Fig*. 1 in the previous example, it can be seen from *Fig*. 3 that the open-loop system in Example 2 is divergent. After adding the designed finite-time bounded controller, it can be seen that the system state changes within the c_2 range within time T in *Fig*. 4. Therefore, the effectiveness of the algorithm in the theorem was demonstrated through simulation.

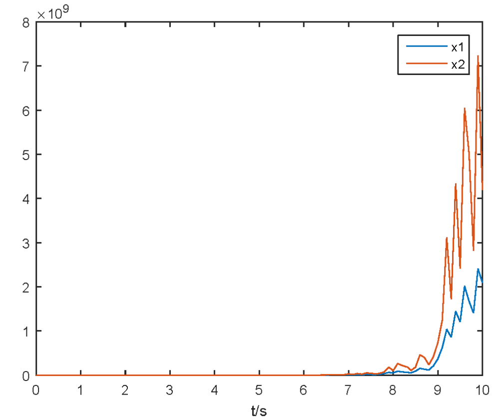

FIGURE 1. The state response of the open-loop system (1)-(3) of example 1.

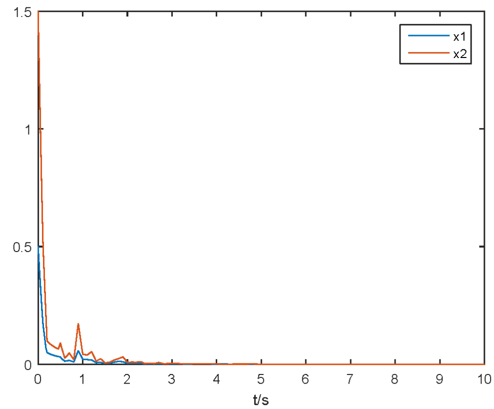

FIGURE 2. The state response of the closed-loop system (12) of example 1.

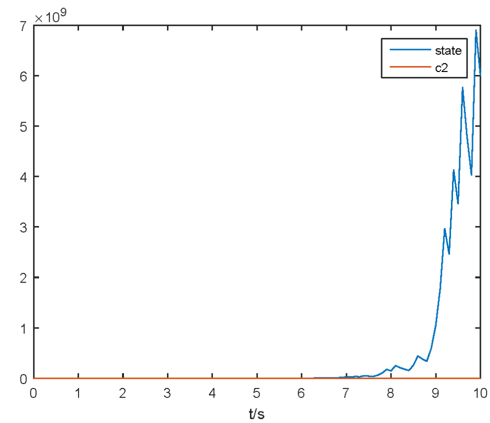

FIGURE 3. $\bar{x}(t)^T \bar{E}^T R_i$ of open-loop system (1)-(3) of example 2.

Remark 3: If we do not consider nonlinear characteristics such as input saturation, from the simulation results the H_∞ performance $\gamma = 0.8$ is less than the results of literature [33].

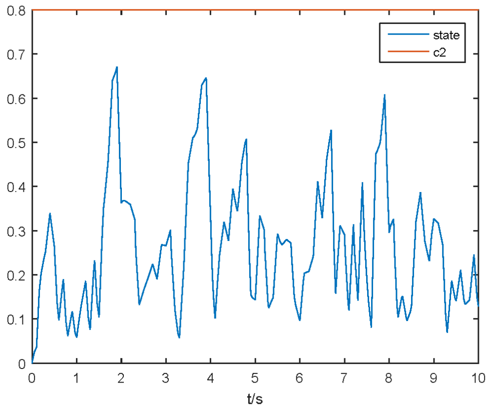

FIGURE 4. $\bar{x}(t)^T\bar{E}^T R_i$ **of closed-loop system (12) of example 2.**

V. CONCLUSION

The article designs a finite-time controller that can use feedback control to achieve H_∞ exponential stability for positive singular Markovian time-delay systems. The theorem presented in the article can be used to calculate the gain matrix of feedback controllers using convex optimization techniques based on linear matrix inequalities, and the effectiveness of the method was validated from the final simulation. The next step of work will be to reduce conservatism and expand the scope of the applicable system of the theorem by relaxing assumptions and considering other unmodeled dynamic characteristics. Compared to other situations such as the uncertainty of state transition rates studied in reference [34], this is also a direction to expand the generality of our work. In addition, other forms of control techniques such as dynamic state feedback and output feedback will be used to develop our control methods.

STATEMENTS AND DECLARATIONS

CONFLICT OF INTEREST

The authors declare that we have no conflicts of interests about the publication of this article.

AVAILABILITY OF DATA AND MATERIAL

The authors declare that all data generated or analyzed during this study are included in this article.

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XUEJING REN received the Ph.D. degree in basic mathematics from Hunan University, Changsha, China, in 2016. She is currently with the School of Science, Changzhou Institute of Technology. Her research interests include complex analysis and dynamic systems.

BO LI received the B.S. degree in electrical engineering and automation from Northwestern Polytechnical University, Xi'an, China, in 2007, and the Ph.D. degree in control science and engineering from Nanjing University of Science and Technology, Nanjing, China, in 2015. He is currently with the Department of Automation, Jiangsu University of Technology. His research interests include robust control and stochastic systems control.

JUNJIE ZHAO received the B.S. degree in automation and the Ph.D. degree in control science and engineering from Nanjing University of Science and Technology, Nanjing, China, in 2008 and 2014, respectively. He joined the Department of Automation, Jiangsu University of Technology, Changzhou, Jiangsu, China, in 2014. His research interest includes control algorithms of nonlinear systems.