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RESEARCH ARTICLE

Distributed Bipartite Output Formation Control for Heterogeneous Discrete-Time Linear Multi-Agent Systems

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ABSTRACT This paper studies the distributed bipartite output formation control problem of heterogeneous discrete-time linear multi-agent systems (MASs) via cooperative output regulation theory. To construct the bipartite formation, under the structurally-balanced augmented directed signed graph, the followers of two antagonistic subgroups are supposed to respectively keep the desired relative positions with the leader, also called as the exosystem, in the same magnitude but the opposite sign. Since the information of exosystem can not be directly obtained by all followers, the distributed exosystem observer based on the discrete-time algebraic Riccati equality (ARE) is presented and the distributed state feedback controller is further designed. Moreover, in the case where the states of followers are not available, the distributed output feedback controller is proposed by introducing the state estimator. Finally, two numerical examples are given to demonstrate the effectiveness of analytic results.

INDEX TERMS Bipartite output formation, discrete-time, multi-agent systems, cooperative output regulation, directed signed graph.

I. INTRODUCTION

The MASs are composed of multiple intelligent and independent agents. In a multi-agent network, the complex collaborative task is completed through the mutual coordination between each single agent. With the rapid development on advanced manufacturing technology and artificial intelligence technology, MASs have received wide attention in many engineering disciplines. Therefore, the cooperative control for the complex MASs has become an important and challenging research topic. For example, cooperative bounded tracking control for multiple unmanned aerial vehicles (UAVs) [1], adaptive formation control for multiple vehicles [2], cooperative surrounding control for multiple robots [3], and so on.

The MASs under cooperative controllers can achieve multiple collective behaviors, such as flocking [4], rendezvous [5], distributed optimization [6], and formation [7]. In particular, formation control has made considerable applications and achievements in military and industrial fields. The main purpose of formation control is to design distributed control protocol to make each agent maintain a predetermined geometric constraint at a certain state level. In recent years, the control community has two main formation control strategies, which are based on the artificial potential field and the consensus theory, respectively. The control strategies

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via artificial potential field [8], [9], [10] mainly use the gradient of local minimum potential function, which have certain advantages in real-time and high efficiency. However, due to the complexity of the artificial potential field, it can only be used to solve the formation control problem of some low-order MASs. Compared with the artificial potential field method, the consensus-based control method is more simpler and flexible, and can be further applied to MASs with complex dynamic models. Recently, consensus problem has become the basis and core of cooperative control problem. In general, the formation control protocol based on consensus theory usually requires a leader who can generate the desired trajectory, and then designs a controller to drive all followers to asymptotically track the leader through a predetermined offset. For example, the simultaneous tracking and formation control problem of MASs was solved in [11] with a virtual leader. The consensus formation control problem of a recurrent neural network was studied in [12].

It should be pointed out that the dynamics of MASs in the above formation control literatures are considered to be homogeneous. In other words, all subsystems have the same dynamic characteristics, which means that the controllers with the same structure cannot meet the control requirements of different subsystems at the same time. Therefore, scholars have begun to pay attention to the formation control problems of heterogeneous MASs in recent years. In particular, a fixed-time formation control protocol for heterogeneous MASs subject to parameter uncertainties, disturbance and actuator faults was presented in [13]. The cooperative formation problem of heterogeneous UAVs with parameter uncertainty was studied in [14]. The time-varying formation of nonlinear heterogeneous multi-agent systems under uncertainty and interference was studied in [15]. However, the control protocols proposed in these literatures are mainly used to achieve state formation control for MASs. Furthermore, how to achieve the consistent output of all agents is also a direction worthy of attention.

The cooperative output regulation principle [16] is an effective method to achieve output cooperative control for heterogenous MASs. Its objective is to make all followers asymptotically track the output trajectory of the autonomous exosystem, which is also viewed as the leader in the classical leader-following MASs. Nowadays, some distributed cooperative formation controllers based on output regulation theory have been proposed in the literatures such as [17] and [18]. However, the above approaches for continuous-time dynamics are not effective for discrete-time case since the discrete stability region is interior of unit circle, which imposes a severe constraint than the continuous-time case with a relaxed and open left-half complex plane. To solve the cooperative output regulation problem of discrete-time MASs, scholars recently proposed the distributed controllers to achieve output consensus of discrete-time counterparts [20], [21]. Note that some contractive conditions are usually needed in these studies. That is, the system matrix of exosystem is neutrally or exponentially stable, which bring difficulty in obtaining the information of exosystem in distributed structure. Thus, how to find an effective observer design method for discrete-time heterogenous MASs still a challenging issue.

In practice, it is difficult to meet the industrial and military requirements if only considering the cooperative relationship among agents. For instance, two groups of UAVs in the opposite direction at the same time [22]. The properties of nonnegative graph in the traditional consensus protocols will not generally hold in the situation that the competition and cooperation interactions coexist in the communication topology with negative weights. Thus, the so-called bipartite consensus was firstly defined in [24] and described in the signed graph with both positive and negative weights, which were used to describe the relationship between cooperation and competition among agents. Moreover, bipartite controller design technique has been applied in the formation control problem under the traditional leader-following framework. The purpose is to make the two competitive subgroups asymptotically keep the desired relative positions with the reference signal of leader in the same magnitude but the opposite sign. For example, the bipartite formation of multi-robot system with time-varying delays was realized in [25]. The event-triggered bipartite consensus of MASs under structural balanced signed graph and its application in satellite formation were realized in [26]. Specially, for the heterogeneous continuous-time MASs, the bipartite output regulation framework was built in [27]. Moreover, the bipartite time-varying formation problem with multiple leaders was solved by using output regulation theory in [28], however, it can not easily be extended to achieve the similar bipartite formation for MASs in discrete time form. There is still lack of a new bipartite output formation framework with heterogeneous and discrete-time dynamics under a general directed signed graph.

Motivated by the above mentioned discussions, we investigate the bipartite output formation control problem of heterogeneous discrete-time linear MASs in this paper, where both cooperative and antagonistic interactions between agents exist in an augmented directed signed graph. First, the bipartite formation issue is converted into the cooperative output regulation problem and the state feedback controller is developed by synthesizing a distributed exosystem observer. Then, considering that the more practical scenarios that the states of followers are not easily accessible, the output feedback controller is constructed by further embedding a state estimator. The salient contributions of this paper are listed as follows:

1) A new output bipartite formation framework of heterogenous discrete-time MASs is established in this paper. Specially, both state and output feedback control laws by virtue of output regulation theory are proposed.

2) A distributed observer based on discrete-time ARE is designed to guarantee the desired relative positions with the

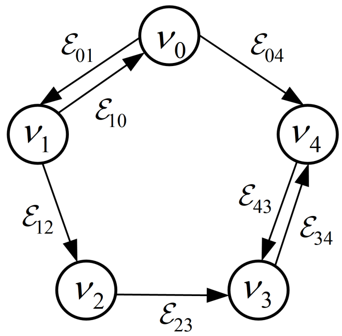

FIGURE 1. **Multi-agent system model diagram.**

exosystem. Moreover, the system matrix of exosystem is no longer constrained to be neutrally or exponentially stable.

3) The proposed distributed bipartite formation controller can be further applied to deal with the similar state/output formation/consensus issues of heteregenous/hotomogenous under directed/undirected nonnegative/signed graphs.

Notations: Through this paper, $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, I_N$ and $\mathbf{1}_N$ represent real number set, complex number set, n-dimensional real vector, N-dimensional unit matrix and N-dimensional column with all elements of 1, respectively. For any real matrix P, P^T and P^{-1} represent transpose matrix and inverse matrix of P, respectively. $\mathrm{diag}(P_1, P_2, \cdots, P_N)$ represents a block diagonal matrix. If P is a symmetric matrix, then $P > 0$ (resp. ≥ 0) means that P is a positive definite (semi-positive definite) matrix, otherwise $P < 0$ (resp. ≤ 0) is negative definite (semi-negative definite) matrix. For the square matrix A, let $\lambda_m(A)$, $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ and $\sigma(A)$ represent the *m-th* eigenvalue, the maximum eigenvalue, the minimum eigenvalue and the spectrum of matrix A, respectively. The symbol $\otimes$ denotes Kronecker product. The sign function $\mathrm{sgn}(\cdot)$ is expresses as

$$\mathrm{sgn}(\cdot) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{1}$$

II. PRELIMINARIES AND PROBLEM STATEMENT

A. PRELIMINARIES

Algebraic graph theory is usually used to analyze the communication topology of MASs, and the corresponding model diagram as shown in Figure 1. $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is used to describe directed signed graph, where $\mathcal{V} = \{v_1, \cdots, v_N\}$ represents the set of nodes composed of N nodes, and each node represents an agent. $\mathcal{E} = \mathcal{V} \times \mathcal{V}$ is defined as an edge set composed of two different nodes, which can also be regarded as information transmission between two agents. $\mathcal{E}_{ji} = (v_j, v_i)$ represents the interaction between node v_i and v_j, that is, node v_j can transmit information to node v_i. Then the neighbor set of node v_i is defined by $\mathcal{N}_i = \{v_j \in \mathcal{V} \,|\, (v_j, v_i) \in \mathcal{E}, i \neq j\}$. Suppose that the directed topology graph $\mathcal{G}$ of this paper is a simple graph and satisfies $(v_i, v_i) \notin \mathcal{E}$, that is, there is no self-loop and multiple edges. $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is defined as the neighbor set matrix of the graph $\mathcal{G}$. If $\mathcal{E}_{ji} \in \mathcal{E}$, one has $a_{ij} > 0$, otherwise $a_{ij} < 0$, where a_{ij} is the weight of edge $\mathcal{E}_{ji}$. Moreover, $a_{ij} > 0$ and $a_{ij} < 0$ represent the cooperative and competitive relationships among agents, respectively.

In traditional nonnegative graphs, the Laplacian matrix plays a fundamental role in the study of consistency problems and can usually be defined as

$$\mathcal{L} = [l_{ij}]_{N\times N} = \mathrm{diag}\left(\sum_{j\in\mathcal{N}_i} a_{1j}, \cdots, \sum_{j\in\mathcal{N}_i} a_{Nj}\right) - \mathcal{A} \tag{2}$$

In this paper, there exists not only cooperation but also competition among each agent. That is, the weights of the corresponding edges of the signed graph can be positive and negative, and the corresponding Laplacian matrix can be defined as

$$\mathcal{L}^s = [\bar{l}_{ij}]_{N\times N} = \mathrm{diag}\left(\sum_{j\in\mathcal{N}_i} |a_{1j}|, \cdots, \sum_{j\in\mathcal{N}_i} |a_{Nj}|\right) - \mathcal{A} \tag{3}$$

Furthermore, $\mathcal{I} = \mathtt{diag}\{\bar{d}_i\} \in \mathbb{R}^{N\times N}$ is defined as an in-degree matrix, where $\bar{d}_i = \sum_{j\in\mathcal{N}_i} |a_{ij}|$. In addition, node v_0 as a leader can exchange information with some followers. Furthermore, by adding the leader into communication topology, an augmented directed signed graph is described by $\widehat{\mathcal{G}}(\widehat{\mathcal{V}}, \widehat{\mathcal{E}}, \widehat{\mathcal{A}})$, where $\widehat{\mathcal{V}} = \{v_0, \cdots, v_N\}$ is the set of nodes. In order to describe the relationship between leader and followers, the pinning matrix is defined as $G = \mathtt{diag}\{g_1, \cdots, g_N\}$. If the node v_i can receive the information transmitted by the leader v_0, then the weight $g_i > 0$, otherwise $g_i = 0$. Specially, if there exists a directed path from the node v_0 to any other nodes in graph $\widehat{\mathcal{G}}$, then $\widehat{\mathcal{G}}$ is said to have a directed spanning tree with v_0 as the root node.

Assumption 1: The directed signed topological graph $\widehat{\mathcal{G}}$ has no multiple edges. Moreover, $\widehat{\mathcal{G}}$ has directed a spanning tree, where v_0 is the leader and root node.

Definition 1 (Structurally Balanced [24]): The signed graph $\widehat{\mathcal{G}}$ is structurally balanced if the node set $\widehat{\mathcal{V}}$ can be divided into two non-empty subsets $\widehat{\mathcal{V}}_1$ and $\widehat{\mathcal{V}}_2$ satisfying $\widehat{\mathcal{V}}_1 \cup \widehat{\mathcal{V}}_2 = \widehat{\mathcal{V}}$ and $\widehat{\mathcal{V}}_1 \cap \widehat{\mathcal{V}}_2 = \varnothing$. For any two nodes $v_i, v_j \in \widehat{\mathcal{V}}_q$ ($q \in \{1, 2\}$) satisfy $a_{ij} \geq 0$ and $v_i \in \widehat{\mathcal{V}}_q$, $v_j \in \widehat{\mathcal{V}}_{3-q}$ satisfy $a_{ij} < 0$. Otherwise, a_{ij} is nonpositive.

Based on the structurally balanced condition given in Definition 1, a diagonal matrix $\mathcal{D} = \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$, $d_{1,2} \in \{1, -1\}$ is constructed to classify two subgroups with competitive relationship. At the same time, in order to solve the bipartite cooperative control problem of multi-agent systems, the following lemma needs to be introduced.

Lemma 1 ([27]): If the signed digraph $\widehat{\mathcal{G}}$ is structurally balanced, then all elements of $\mathcal{DAD}$ are nonnegative. Moreover, if the directed spanning tree assumption is satisfied, then the matrices $\overline{\mathcal{L}}^s = \mathcal{L}^s + G$ and $L = \mathcal{D}(\mathcal{L}^s + G)\mathcal{D} = \mathcal{D}\overline{\mathcal{L}}^s\mathcal{D}$ are positive definite.

Lemma 2 (Geršgorin Circle Criterion [29]): All eigenvalues of matrix $\Pi = \left[\vartheta_{ij}\right] \in \mathbb{R}^{N\times N}$ are in the union of the following N disks.

$$\bigcup_{i=1}^{N}\left\{s \in \mathbb{C} : |s-\vartheta_{ii}| \le \sum_{j\neq 1, i=i}^{N}\left|\vartheta_{ij}\right|\right\} \tag{4}$$

B. PROBLEM FORMULATION

We consider a discrete-time linear MAS which is composed by a group of N nonidentical followers and one leader. The network topology associated with the MAS is represented as the signed digraph $\widehat{\mathcal{G}}$. The dynamics of followers can be governed by

$$\begin{cases} x_i(t+1) = & A_i x_i(t) + B_i u_i(t) \\ y_i(t) = & C_i x_i(t) + D_i u_i(t), \ i = 1, \cdots, N, \ t \in \mathbb{Z}^+, \end{cases} \tag{5}$$

where the vectors $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^p$ respectively denote the state, the control input, and the measurement output of follower agents in two different subgroups.

The reference trajectory is generated by the leader, which has an automatous discrete-time linear dynamics and is governed by

$$\begin{cases} r(t+1) = & S_r r(t) \\ y_r(t) = & C_r r(t), \quad r(0) = r_0 \end{cases} \tag{6}$$

where the vectors $r(t) \in \mathbb{R}^h$ and $y_r(t) \in \mathbb{R}^p$ respectively denote the state and the measurement output of leader.

To construct the desired bipartite formation, the relative positions are given as

$$\lim_{t\to\infty}(y_i(t) - d_i y_r(t)) = \delta_i(t), \ i = 1, \cdots, N \tag{7}$$

where d_i is the element of matrix $\mathcal{D}$ and $\delta_i(t) \in \mathbb{R}^p$ is a constant vector. Therefore, the desired relative position between followers i and j is denoted as

$$\begin{aligned} \delta_{ij}(t) &= y_i(t) - y_j(t) \\ &= (d_i y_r(t) - d_j y_r(t)) + (\delta_i(t) - \delta_j(t)) \end{aligned} \tag{8}$$

In particular, the vector $\delta_{ij}(t)$ between two followers in the same subgroup is simplified as $\delta_{ij}(t) = \delta_i(t) - \delta_j(t)$.

Hence, the bipartite formation error vectors are defined as follows:

$$e_i(t) = y_i(t) - d_i y_r(t) - \delta_i(t), \ i = 1, \cdots, N \tag{9}$$

Note that if $\lim_{t\to\infty} e_i(t) = 0, i = 1, \cdots, N$ are satisfied then the so-called bipartite formation can be achieved.

To achieve the bipartite formation control, the output regulation theory is introduced in our paper. Under this framework, exogenous reference signal is expressed as

$$y_{vi}(t) \triangleq d_i y_r(t) + \delta_i(t) \tag{10}$$

Define

$$v_i(t) = \left(d_i r^T(t), \delta_i^T(t)\right)^T$$
$$S_v = \begin{pmatrix} S_r & 0 \\ 0 & I_p \end{pmatrix}$$
$$C_v = (C_r, I_p)$$

Then the so-called exosystem which can generate bipartite reference signal of each follower can be written as

$$\begin{cases} v_i(t+1) = S_v v_i(t) \\ y_{vi}(t) = C_v v_i(t) \end{cases}, \ i = 1, \cdots, N \tag{11}$$

Combine with the output regulation technique, the so-called bipartite formation control problem is described as follows:

Consider a heterogeneous discrete-time linear MAS composed by (5) and (6), the bipartite formation control can be achieved by designing the distributed controllers u_i, $i = 1, \cdots, N$, i.e.,

$$\lim_{t\to\infty} e_i(t) = 0, \ i = 1, \cdots, N \tag{12}$$

Remark 1: Different form the traditional formation control issues based upon the state consensus strategy over nonnegative graphs, in this paper, the control aim is that make all the outputs of followers in two subgroups with antagonistic interactions track the leader in the same magnitude but the opposite sign, while keeping the desired relative positions with the exosystem. Thereby, the term (12) can be equivalently written as

$$\begin{cases} \lim_{t\to\infty}(y_i(t) - \delta_i(t)) = & y_r(t), \quad \forall i \in \mathcal{V}_1 \\ \lim_{t\to\infty}(y_i(t) - \delta_i(t)) = & -y_r(t), \quad \forall j \in \mathcal{V}_2 \end{cases} \tag{13}$$

III. MAIN RESULTS

In this section, two classes of distributed control protocols are presented to accomplish the bipartite formation task, and the corresponding control scheme is given in Figure 2. First, since the reference information for the exosystem (11) is not obtained by all followers, a new distributed observer based upon the discrete-time standard ARE is introduced. Then considering the case where states of followers are not available, the state estimator is introduced and the distributed bipartite formation control law under output feedback is further designed. The following assumptions and lemma are needed to realize the bipartite formation control.

Assumption 2: The pairs $(A_i, B_i), i = 1, \cdots, N$ are stabilisable.

Assumption 3: The pairs $(A_i, C_i), i = 1, \cdots, N$ are detectable.

Assumption 4: $\mathrm{Re}(\lambda_i(S_r)) \ge 1, i = 1, \cdots, h$ holds, that is, all eigenvalues of exosystem matrix S_r locate outside the unit circle in the z-plane.

Assumption 5: For all $\lambda \in \sigma(S_v)$,

$$\operatorname{rank}\begin{pmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{pmatrix} = n_i + p \tag{14}$$

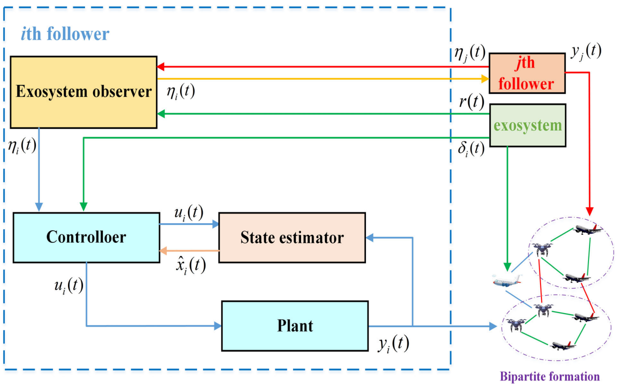

FIGURE 2. **Control scheme of bipartite output formation.**

Lemma 3 ([16]): Condition (14) is also called the transmission zero condition in the cooperative output regulation problem. If Assumption 5 holds, then the output regulation equations

$$\begin{aligned} X_iS_v &= A_iX_i + B_iU_i \\ 0 &= C_iX_i + D_iU_i - C_v \end{aligned} \tag{15}$$

have unique solution pair (X_i, U_i).

Remark 2: Assumptions 2 and 3 provide the stabilisable and detectable conditions for the closed-loop system, respectively. Note that the matrix S_v is a block diagonal matrix with elements S_r and I_p, so part of eigenvalues of S_v are the same as the eigenvalues of S_r, and others are 1. Therefore, Assumption 4 can make sure that all eigenvalues of S_v locate outside the unit circle. Assumption 5 guarantees that the output regulation equations have unique solution pairs by Lemma 3.

A. STATE FEEDBACK FORMATION CONTROLLER DESIGN

In many practical applications, the information of exosystem is not available to all followers. Thus, a distributed exosystem observer is embedded in the control architecture. Based on the state feedback, the bipartite formation control law is designed as follows:

$$\begin{cases} u_i(t) = K_{1i}x_i(t) + K_{2i}\eta_i(t) + K_{3i}\delta_i(t) \\ \eta_i(t+1) = S_r\eta_i(t) - \mu_iFH\xi_i(t) \\ \xi_i(t) = \sum\limits_{j\in\mathcal{N}_i} \left|a_{ij}\right| (\eta_i(t) - \mathrm{sgn}(a_{ij})\eta_j(t)) \\ +g_i(\eta_i(t) - d_ir(t)) \end{cases} \tag{16}$$

where $K_{1i} \in \mathbb{R}^{m_i\times n_i}$, $K_{2i} \in \mathbb{R}^{m_i\times h}$ and $K_{3i} \in \mathbb{R}^{m_i}$ are the gain matrices. $\eta_i(t) \in \mathbb{R}^h$ is the reference signal observation of exosystem. μ_i is the constant coupling observer gain. $F \in \mathbb{R}^{h\times p}$ and $H \in \mathbb{R}^{p\times h}$ are the observer gain matrices. g_i and d_i are the elements of matrices G and $\mathcal{D}$.

The following lemmas need to be utilized in the convergence analysis for the distributed exosystem observer.

Lemma 4 ([21]): If Q is a real positive symmetric matrix, then it can be decomposed into $Q = Q_dQ_d$, where $Q_d = Q_d^T > 0$ is also a real symmetric matrix.

Lemma 5: For the detectable matrix pair (S_r, H), select the gain as

$$F = S_rP_1H^T(HP_1H^T + R_1)^{-1} \tag{17}$$

where $P_1 = P_1^T > 0$ and $R_1 = R_1^T > 0$. If $s \in \mathbb{C}$ is located in the following stability region

$$\bar{\Psi} = \left\{ s \in \mathbb{C} \,\middle|\, |s-1|^2 < \frac{1}{\bar{Q}_d^{-1}W_s\bar{Q}_d} \right\}, \tag{18}$$

then all the eigenvalues of $S_r - sFH$, $s \in \mathbb{C}$ are Schur, where $\bar{Q}_d = \bar{Q}_d^T > 0$ and $W_s = S_rP_1^TH(HP_1^TH + R_1)^{-1}HP_1S_r^T$.

Proof: If there exists the detectable matrix pair (S_r, H), then the Schur stability of matrix $S_r - sFH, s \in \mathbb{C}$ can be guaranteed. By [30], the following standard discrete-time ARE

$$S_rP_1S_r^T - P_1 - W_s + Q_s = 0 \tag{19}$$

has a positive symmetric definite matrix solution $P_1 = P_1^T > 0$, where $Q_s = Q_s^T > 0$. By using the Lyapunov stability theorem, if

$$\bar{\Xi} = (S_r - sFH)P_1(S_r - sFH)^* - P_1 < 0, \tag{20}$$

then the matrix $S_r - sFH$ is Schur. Let s =Re(s)+Im(s)j and F as (17), then one has

$$\begin{aligned}
\bar{\Xi} &= (S_r - sFH)P_1(S_r - sFH)^* - P_1 \\
&= S_rP_1S_r^T - (\mathrm{Re}(s) + \mathrm{Im}(s)j)FHP_1S_r^T \\
&\quad - (\mathrm{Re}(s) - \mathrm{Im}(s)j)S_rP_1H^TF^T \\
&\quad + (\mathrm{Re}^2(s) + \mathrm{Im}^2(s))FHP_1H^TF^T - P_1 \\
&= S_rP_1S_r^T - (\mathrm{Re}(s) + \mathrm{Im}(s)j)S_rP_1H^T(HP_1H^T + R_1)^{-1} \\
&\quad \cdot HP_1S_r^T - (\mathrm{Re}(s) - \mathrm{Im}(s)j)S_rP_1H^T(HP_1H^T + R_1)^{-1} \\
&\quad \cdot HP_1S_r^T + (\mathrm{Re}^2(s) + \mathrm{Im}^2(s))S_rP_1H^T(HP_1H^T + R_1)^{-1} \\
&\quad \cdot HP_1H^T(HP_1H^T + \check{R})^{-1}HP_1S_r^T - P_1 \\
&= S_rP_1S_r^T - 2\mathrm{Re}(s)S_rP_1H^T(HP_1H^T + R_1)^{-1}HP_1S_r^T \\
&\quad + (\mathrm{Re}^2(s) + \mathrm{Im}^2(s))S_rP_1H^T(HP_1H^T + R_1)^{-1} \\
&\quad \cdot (HP_1H^T + R_1 - R_1)(HP_1H^T + R_1)^{-1}HP_1S_r^T - P_1 \\
&< S_rP_1S_r^T - P_1 - (\mathrm{Re}^2(s) + \mathrm{Im}^2(s) - 2\mathrm{Re}(s)) \\
&\quad \cdot S_rP_1H^T(HP_1H^T + R_1)^{-1}HP_1S_r^T - (\mathrm{Re}^2(s) + \mathrm{Im}^2(s)) \\
&\quad \cdot R_1(HP_1H^T + R_1)^{-1}HP_1S_r^T \\
&< S_rP_1S_r^T - P_1 + (|s|^2 - 2\mathrm{Re}(s)) \\
&\quad \cdot S_rP_1H^T(HP_1H^T + R_1)^{-1}HP_1S_r^T \\
&= S_r^TP_1S_r - P_1 - S_rP_1H^T(HP_1H^T + R_1)^{-1}HP_1S_r^T \\
&\quad + (|s|^2 - 2\mathrm{Re}(s) + 1)S_rP_1H^T(HP_1H^T + R_1)^{-1}HP_1S_r^T \\
&< -Q_s + |s-1|^2\, W_s
\end{aligned} \tag{21}$$

From Lemma 4, the positive symmetric matrix Q_s can be decomposed into $Q_s = \bar{Q}_d\bar{Q}_d$, where $\bar{Q}_d = \bar{Q}_d^T > 0$.

Correspondingly, the inequation

$$\begin{aligned} -Q_s + |s-1|^2 W_s &= \bar{Q}_d \bar{Q}_d + |s-1|^2 W_s \\ &< 0 \end{aligned} \tag{22}$$

is equivalent to

$$-I + |s-1|^2 \bar{Q}_d^{-1} W_s \bar{Q}_d < 0 \tag{23}$$

by using the premultiplication and postmultiplication with $\bar{Q}_d^{-1}$ and $\bar{Q}_d$. It is obvious that if (22) or (23) holds, then $\bar{\Xi} < 0$ is satisfied. Therefore, by (23), we can conclude that if

$$|s-1|^2 < \frac{1}{\bar{Q}_d^{-1} W_s \bar{Q}_d} \tag{24}$$

then the matrix $S_r - sFH$ is Schur stable, which accomplishes the proof. ∎

Next, we will give the main theorem of this subsection under the distributed bipartite formation state feedback controller (16).

Theorem 1: Under the directed signed graph $\widehat{\mathcal{G}}$, a heterogeneous discrete-time linear MAS is consisting of (5) and (6), which satisfies Assumptions 1, 2, 4 and 5. The corresponding bipartite formation control can be achieved by using the distributed state feedback control law (16), if the control parameters are designed to satisfy the below two conditions:

1) The observer gain μ_i is designed as $\mu_i = 1/(\bar{l}_{ii}+g_i)$. The observer gain matrices F and H are designed from Lemma 5, which are associated with the discrete-time ARE (19).

2) The gain matrix K_{1i} is designed to make $A_i + B_i K_{1i}$ is Schur. The gain matrices K_{2i} and K_{3i} are selected as $(K_{2i}, K_{3i}) = U_i - K_{1i} X_i$.

Proof: Define the following compact vectors

$$\begin{aligned} x(t) &= \left(x_1^T(t), \cdots, x_N^T(t)\right)^T \ \eta(t) = \left(\eta_1^T(t), \cdots, \eta_N^T(t)\right)^T \\ \delta(t) &= \left(\delta_1^T(t), \cdots, \delta_N^T(t)\right)^T \ v(t) = \left(v_1^T(t), \cdots, v_N^T(t)\right)^T \\ e(t) &= \left(e_1^T(t), \cdots, e_N^T(t)\right)^T \ \mu = \left(\mu_1^T, \cdots, \mu_N^T\right)^T \\ x_c(t) &= \left(x^T(t), \eta^T(t)\right)^T \end{aligned} \tag{25}$$

and following compact matrices

$$\begin{aligned} A &= \mathrm{diag}\,(A_1, \cdots, A_N) \ B = \mathrm{diag}\,(B_1, \cdots, B_N) \\ C &= \mathrm{diag}\,(C_1, \cdots, C_N) \ D = \mathrm{diag}\,(D_1, \cdots, D_N) \\ K_1 &= \mathrm{diag}\,(K_{11}, \cdots, K_{1N}) \ K_2 = \mathrm{diag}\,(K_{21}, \cdots, K_{2N}) \\ K_3 &= \mathrm{diag}\,(K_{31}, \cdots, K_{3N}) \end{aligned} \tag{26}$$

The closed-loop system under the bipartite formation controller (16) can be written as

$$\begin{cases} x_c(t+1) = & A_c x_c(t) + \begin{pmatrix} BK_3\delta(t) \\ (\mathcal{G}\mathcal{D} \otimes FH)(\mathbf{1}_N \otimes r(t)) \end{pmatrix} \\ e(t) = & C_c x_c(t) + DK_3\delta(t) - (I_N \otimes C_v)v(t) \end{cases} \tag{27}$$

where

$$\begin{aligned} A_c &= \begin{pmatrix} A + BK_1 & BK_2 \\ 0 & I_N \otimes S_r - \mu\overline{\mathcal{L}}^s \otimes FH \end{pmatrix} \\ C_c &= (C + DK_1, DK_2) \end{aligned} \tag{28}$$

Next, we will give two statements to prove the bipartite formation result under the controller (16).

Statement 1): It should be noted that the exosystem observer with compact form plays an essential role in the stability of dynamics (27), which is described as follows:

$$\begin{aligned} \eta(t+1) = &(I_N \otimes S_r - \mu\overline{\mathcal{L}}^s \otimes FH)\eta(t) \\ &+ (\mathcal{G}\mathcal{D} \otimes FH)(\mathbf{1}_N \otimes r(t)) \end{aligned} \tag{29}$$

To analyze the stability of system (29), define the following error vector and its compact form

$$\begin{aligned} \tilde{\eta}_i(t) &= \eta_i(t) - d_i r(t) \\ \tilde{\eta}(t) &= \left(\tilde{\eta}_1^T(t), \cdots, \tilde{\eta}_N^T(t)\right)^T \end{aligned} \tag{30}$$

Under Assumption 1, any two connected followers in directed signed graph $\widehat{\mathcal{G}}$ have only one interaction relation, then we have $a_{ij}d_id_j \geq 0$ for $i, j \in \mathcal{V}$ and $d_i \in \{\pm 1\}$. Therefore, one has the following equivalence relations

$$\begin{cases} \left|a_{ij}\right| = a_{ij}\mathrm{sgn}(a_{ij}) \\ \left|a_{ij}\right| d_j = a_{ij} d_i \end{cases} \tag{31}$$

Based on (30) and (31), the difference of $\tilde{\eta}_i(t)$ can be calculated as follows:

$$\begin{aligned} &\tilde{\eta}_i(t+1) \\ &= \eta_i(t+1) - d_i r(t+1) \\ &= S_r\eta_i(t) - \mu_i FH(\sum_{j\in\mathcal{N}_i} \left|a_{ij}\right| (\eta_i(t) - \mathrm{sgn}(a_{ij})\eta_j(t)) \\ &\quad + g_i(\eta_i(t) - d_i r(t))) - d_i S_r r(t) \\ &= S_r(\eta_i(t) - d_i r(t)) - \mu_i FH(\sum_{j\in\mathcal{N}_i} \left|a_{ij}\right| (\eta_i(t) - d_i r(t)) \\ &\quad - \left|a_{ij}\right| (\mathrm{sgn}(a_{ij})\eta_j(t) - d_i r(t)) + g_i(\eta_i(t) - d_i r(t))) \\ &= S_r(\eta_i(t) - d_i r(t)) - \mu_i FH(\sum_{j\in\mathcal{N}_i} \left|a_{ij}\right| (\eta_i(t) - d_i r(t)) \\ &\quad - \left|a_{ij}\right| \mathrm{sgn}(a_{ij})(\eta_j(t) - d_j r(t)) + g_i(\eta_i(t) - d_i r(t))) \\ &= S_r\tilde{\eta}_i(t) \\ &\quad - \mu_i FH(\sum_{j\in\mathcal{N}_i} \left|a_{ij}\right| (\tilde{\eta}_i(t) - \mathrm{sgn}(a_{ij})\tilde{\eta}_j(t)) + g_i\tilde{\eta}_i(t)) \end{aligned} \tag{32}$$

and the compact form of dynamics (32) is

$$\tilde{\eta}(t+1) = (I_N \otimes S_r - \mu\bar{\mathcal{L}}_s \otimes FH)\tilde{\eta}(t) \tag{33}$$

Introduce the following compact vector

$$\bar{\eta}(t) = (\bar{\eta}_1^T(t), \cdots, \bar{\eta}_N^T(t))^T \quad = (T^{-1} \otimes I_h)\tilde{\eta}(t) \tag{34}$$

where $T \in \mathbb{R}^{N\times N}$ is the nonsingular matrix satisfying $T^{-1}(\mu\bar{\mathcal{L}}_s)T = J_s$. We can further have the dynamics as follows:

$$\begin{aligned}\bar{\eta}(t+1) &= (T^{-1}\otimes I_h)(I_N\otimes S_r-\mu\bar{\mathcal{L}}_s\otimes FH)(T\otimes I_h)\bar{\eta}(t)\\ &= (I_N\otimes S_r-T^{-1}(\mu\bar{\mathcal{L}}_s)T\otimes FH)\bar{\eta}(t)\\ &= (I_N\otimes S_r-J_s\otimes FH)\bar{\eta}(t)\end{aligned}\tag{35}$$

Obviously, the systems (35) and (33) have the same asymptotic stability. Moreover, if the following matrix

$$I_N\otimes S_r-J_s\otimes FH\tag{36}$$

is Schur, then the stability of (35) can be guaranteed. Due to (36) has an upper triangular structure, we can find that if the eigenvalues of matrices

$$S_r-\varphi_i FH,\ i=1,\cdots,N\tag{37}$$

are Schur, then the matrix (36) is also Schur, where φ_i, $i=1,\cdots,N$ are the eigenvalues of $\mu\bar{\mathcal{L}}_s$. By Lemma 2, φ_i, $i=1,\cdots,N$ will locate in the following unions of N discs

$$\bigcup_{i=1}^{N}\left\{s\in\mathbb{C}:\left|s-\mu_i(\bar{l}_{ii}+g_i)\right|\le\mu_i\bar{l}_{ii}\right\}\tag{38}$$

If the observer gain $\mu_i=1/(\bar{l}_{ii}+g_i)$, then (38) can be rewritten as

$$\bigcup_{i=1}^{N}\left\{s\in\mathbb{C}:|s-1|\le\frac{\bar{l}_{ii}}{\bar{l}_{ii}+g_i}\right\}\tag{39}$$

By introducing the gain μ_i, we can find that all the eigenvalues of $\mu\bar{\mathcal{L}}_s$ can locate in the unit circle. From Lemma 5, the matrix $S_r-\varphi_i FH$ is Schur if φ_i locates in the stability region (18) when H is selected such that (S_r,H) is Schur and F is designed based upon the discrete-time ARE (19). Therefore, if $\sigma(S_v)\subset\bar{\Psi}$ holds, then the matrices $S_r-\varphi_i FH$ and $I_N\otimes S_r-J_s\otimes FH$ are both Schur. Accordingly, the systems (33) and (35) are Schur stable, which also shows that

$$\lim_{t\to\infty}\tilde{\eta}_i(t)=\lim_{t\to\infty}(\eta_i(t)-d_i r(t))=0,\ \ i=1,\cdots,N\tag{40}$$

and the distributed observer can realize the bipartite tracking for the exosystem.

Statement 2): The stability of system (27) can be transformed into that of the closed-loop system composing of error vectors. Let

$$\tilde{x}_i(t)=x_i(t)-X_i v_i(t)\tag{41}$$

where X_i is related with the regulator equations (15). Then the compact vector of (41) is

$$\tilde{x}(t)=\left(\tilde{x}_1^T(t),\cdots,\tilde{x}_N^T(t)\right)^T\tag{42}$$

Define the following compact vector

$$\begin{aligned}\tilde{x}_c(t) &= \left(\tilde{x}^T(t),\tilde{\eta}^T(t)\right)^T\\ &= \left(\tilde{x}_1^T(t),\cdots,\tilde{x}_N^T(t),\tilde{\eta}_1^T(t),\cdots,\tilde{\eta}_N^T(t)\right)^T\end{aligned}\tag{43}$$

and following compact matrices

$$\begin{aligned}X &= \mathrm{diag}\,(X_1,\cdots,X_N)\\ K_{23} &= \mathrm{diag}\,((K_{21},K_{31}),\cdots,(K_{2N},K_{3N}))\end{aligned}\tag{44}$$

Then the system (27) can be rewritten as the following error system

$$\begin{cases}\tilde{x}_c(t+1)=A_c\tilde{x}_c(t)+B_c v(t)\\ e(t)=C_c\tilde{x}_c(t)+D_c v(t)\end{cases}\tag{45}$$

where

$$\begin{aligned}B_c &= \begin{pmatrix}-X(I_N\otimes S_v)+(A+BK_1)X+BK_{23}\\ 0\end{pmatrix}\\ D_c &= (C+DK_1)X+DK_{23}-I_N\otimes C_v\end{aligned}\tag{46}$$

If $(K_{2i},K_{3i})=U_i-K_{1i}X_i$ in condition 2) holds, then the regulator equations (15) are rewritten as

$$\begin{aligned}X_iS_v &= (A_i+B_iK_{1i})X_i+B_i(K_{2i},K_{3i})\\ 0 &= (C_i+D_iK_{1i})X_i+D_i(K_{2i},K_{3i})-C_v\end{aligned}\tag{47}$$

Thus, if the first-line equation of (47) is satisfied, then $B_c=0$ and it can be found that the stability of dynamics $\tilde{x}_c(t)$ only depends on the A_c, which has the block upper triangular form. Since there exists the gain matrices K_{1i} so that $A_i+B_iK_{1i}$ is Schur under Assumption 2, the diagonal block elements of A_c are Schur. One can conclude that

$$\lim_{t\to\infty}\tilde{x}_i(t)=\lim_{t\to\infty}(x_i(t)-X_iv_i(t))=0,\ \ i=1,\cdots,N\tag{48}$$

On the other hand, if the second-line equation of (47) is satisfied, then $D_c=0$ and we can further obtain

$$\lim_{t\to\infty}e_i(t)=\lim_{t\to\infty}(y_i(t)-d_iy_r(t)-\delta_i(t))=0,\ \ i=1,\cdots,N\tag{49}$$

On account of the statements 1) and 2), the distributed bipartite output formation control problem via state feedback can be achieved based on the output regulation theory. The proof is completed. ■

B. OUTPUT FEEDBACK FORMATION CONTROLLER DESIGN

The state feedback controller (16) can realize the bipartite formation by utilizing the state and estimation of exosystem. However, the state of agent is difficult to be obtained in practice, especially in large-scale network structure. In this subsection, the bipartite formation control law is further designed via output feedback by introducing the state estimator, which has the following form

$$\begin{cases}u_i(t)=K_{1i}\hat{x}_i(t)+K_{2i}\eta_i(t)+K_{3i}\delta_i(t)\\ \hat{x}_i(t+1)=A_i\hat{x}_i(t)+B_iu_i(t)-\gamma_i\Gamma_i\tilde{y}_i(t)\\ \eta_i(t+1)=S_r\eta_i(t)-\mu_iFH\xi_i(t)\\ \xi_i(t)=\sum\limits_{j\in\mathcal{N}_i}\left|a_{ij}\right|(\eta_i(t)-sgn(a_{ij})\eta_j(t))\\ +g_i(\eta_i(t)-d_ir(t))\end{cases}\tag{50}$$

where $\hat{x}_i(t) \in \mathbb{R}^{n_i}$ is the state estimation, γ_i is the constant coupling estimator gain and $\Gamma_i \in \mathbb{R}^{n_i \times p}$ is the estimator gain matrix. $\tilde{y}_i(t) = \hat{y}_i(t) - y_i(t) = C_i(\hat{x}_i(t) - x_i(t))$. K_{1i}, K_{2i}, K_{3i}, μ_i, F and H are also the control parameters, which can be similarly designed from the distributed bipartite formation controller (16).

The following lemma will be used in the convergence analysis for the state estimator.

Lemma 6: For the detectable matrix pair (A_i, C_i), select the gain as

$$\Gamma_i = A_i P_2^T C_i (C_i P_2^T C_i + R_2)^{-1} \tag{51}$$

where $P_2 = P_2^T > 0$ and $R_2 = R_2^T > 0$. If $s \in \mathbb{C}$ is located in the following stability region

$$\hat{\Psi} = \left\{ s \in \mathbb{C} \,\middle|\, |s-1|^2 < \frac{1}{\hat{Q}_d^{-1} \hat{W}_s \hat{Q}_d} \right\}, \tag{52}$$

then all the eigenvalues of $A_i - s\Gamma_i C_i$, $s \in \mathbb{C}$ are Schur, where $\hat{Q}_d = \hat{Q}_d^T > 0$ and $\hat{W}_s = A_i P_2^T C_i (C_i P_2^T C_i + R_2)^{-1} C_i P_2^T A_i^T$.

Proof: Under Assumption 2, there exists the detectable matrix pair (A_i, C_i), then the Schur stability of matrix $A_i - s\Gamma_i C_i$, $s \in \mathbb{C}$ can be guaranteed. Moreover, the following standard discrete-time ARE

$$A_i P_1 A_i^T - P_2 - \hat{W}_s + \hat{Q}_s = 0 \tag{53}$$

has a positive symmetric definite matrix solution $P_2 = P_2^T > 0$, where $\hat{Q}_s = \hat{Q}_s^T > 0$. By using the Lyapunov stability theorem, if

$$\hat{\Xi} = (A_i - s\Gamma_i C_i) P_2 (A_i - s\Gamma_i C_i)^* - P_2 < 0, \tag{54}$$

then the matrix $A_i - s\Gamma_i C_i$ is Schur. Let $s = \mathrm{Re}(s) + \mathrm{Im}(s)j$ and Γ_i as (51), then we have

$$\begin{aligned}
\hat{\Xi} &= (A_i - s\Gamma_i C_i) P_2 (A_i - s\Gamma_i C_i)^* - P_2 \\
&< A_i P_2 A_i^T - P_2 - (\mathrm{Re}^2(s) + \mathrm{Im}^2(s) - 2\mathrm{Re}(s)) \\
&\quad \cdot A_i P_2^T C_i (C_i P_1 C_i^T + R_1)^{-1} C_i P_2 A_i^T - (\mathrm{Re}^2(s) + \mathrm{Im}^2(s)) \\
&\quad \cdot R_1 (C_i P_2 C_i^T + R_2)^{-1} C_i P_2 A_i^T \\
&< A_i P_2 A_i^T - P_2 + (|s|^2 - 2\mathrm{Re}(s)) \\
&\quad \cdot A_i P_2^T C_i^T (C_i P_2 C_i^T + R_2)^{-1} C_i P_2 A_i^T \\
&= A_i^T P_2 A_i - P_2 - A_i P_2 C_i^T (C_i P_1 C_i^T + R_2)^{-1} C_i P_2 A_i^T \\
&\quad + (|s|^2 - 2\mathrm{Re}(s) + 1) A_i P_2 C_i^T (C_i P_2 C_i^T + R_2)^{-1} C_i P_2 A_i^T \\
&< -\hat{Q}_s + |s-1|^2 \hat{W}_s
\end{aligned} \tag{55}$$

Obviously, by Lemma 4, $\hat{Q}_s$ can also be decomposed into $\hat{Q}_s = \hat{Q}_d \hat{Q}_d$, where $\hat{Q}_d = \hat{Q}_d^T > 0$. Thus, if

$$-I + |s-1|^2 \hat{Q}_d^{-1} \hat{W}_s \hat{Q}_d < 0 \tag{56}$$

then $\hat{\Xi} < 0$ holds. The rest of the proof is similar with that of lemma 5, and thus is omitted here. ■

Theorem 2: Under the directed signed graph $\widehat{\mathcal{G}}$, a heterogeneous discrete-time linear MAS is consisting of (5) and (6), which satisfies Assumptions 1-5. The corresponding bipartite formation control can be achieved by using the distributed output feedback control law (50), if the control parameters are designed to satisfy the below two conditions:

1) The observer parameters are designed as Theorem 1. The estimator gain γ_i is select to locate in the stability region (52). The estimator gain matrices Γ is designed from Lemma 6, which are associated with the discrete-time ARE (53).

2) The gain matrices K_{1i}, K_{2i} and K_{3i} are selected as Theorem 1.

Proof: The second-line equation of (50) is the state estimator to obtain the information of state based upon the neighbor information. To analyze the convergence of state estimator, define the following error vector

$$\tau_i(t) = \hat{x}_i(t) - x_i(t) \tag{57}$$

which has the compact form as follows:

$$\tau(t) = \left(\tau_1^T(t), \cdots, \tau_N^T(t)\right)^T \tag{58}$$

Then the dynamics of $\tau_i(t)$ is calculated as

$$\begin{aligned}
\tau_i(t+1) &= \hat{x}_i(t+1) - x_i(t+1) \\
&= A_i \hat{x}_i(t) + B_i u_i(t) - \gamma_i \Gamma_i \tilde{y}_i(t) \\
&\quad - (A_i x_i(t) + B_i u_i(t)) \\
&= A_i(\hat{x}_i(t) - x_i(t)) - \gamma_i \Gamma_i (C_i(\hat{x}_i(t) - x_i(t)) \\
&= (A_i - \gamma_i \Gamma_i C_i) \tau_i(t)
\end{aligned} \tag{59}$$

whose asymptotic stability depend on the matrix $A_i - \gamma_i \Gamma_i C_i$. By the detectable condition in Assumption 2 and Lemma 6, the matrix $A_i - \gamma_i \Gamma_i C_i$ is Schur if γ_i locates in the stability region (52) when Γ_i is designed based upon the discrete-time ARE (53). Therefore, if condition 1) in Theorem 2 holds, then the matrices $A_i - \gamma_i \Gamma_i C_i$, $i = 1, \cdots$ are Schur stable and one can conclude that

$$\lim_{t\to\infty} \tau_i(t) = \lim_{t\to\infty} (\hat{x}_i(t) - x_i(t)) = 0, \; i = 1, \cdots, N \tag{60}$$

Define

$$\begin{aligned}
\chi_c &= \left(x_c^T(t), \tau^T(t)\right)^T \; \gamma = \left(\gamma_1^T, \cdots, \gamma_N^T\right)^T \\
\Gamma &= \mathrm{diag}\left(\Gamma_1^T(t), \cdots, \Gamma_N^T(t)\right)^T
\end{aligned} \tag{61}$$

Next, we are ready to construct the closed-loop error system as follows:

$$\begin{cases} \chi_c(t+1) = \bar{A}_c \chi(t) + \bar{B}_c v(t) \\ e(t) = \bar{C}_c \chi(t) + D_c v(t) \end{cases} \tag{62}$$

where

$$\begin{aligned}
\bar{A}_c &= \begin{pmatrix} A_c & -BK_1 \\ 0 & A - \gamma\Gamma C \end{pmatrix} \qquad \bar{B}_c = \begin{pmatrix} B_c \\ 0 \end{pmatrix} \\
\bar{C}_c &= (C_c, -(C + DK_1))
\end{aligned} \tag{63}$$

and D_c is given in (46). If condition 2) in Theorem 2 holds, then the regulator equations (47) is satisfied and we have $\bar{B}_c = 0$ and $D_c = 0$. Therefore, the stability of $\chi_c(t)$ is

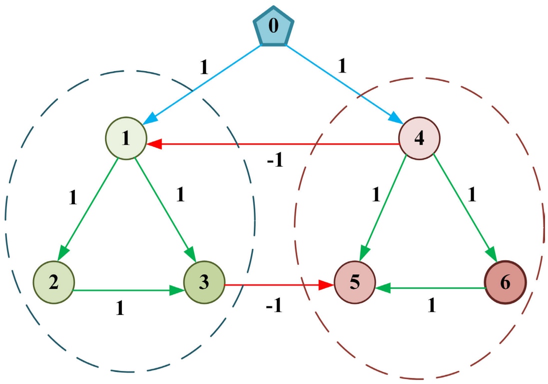

FIGURE 3. Topology of the multi-vehicle system with two connected subgroups.

determined by the block upper triangular matrix $\bar{A}_c$, which is governed by A_c and $A - \gamma\Gamma C$. The stability of A_c is illustrated in Theorem 1. The matrix $A - \gamma\Gamma C$ is diagonally composed by the Schur stable matrices $A_i - \gamma_i \Gamma_i C_i, i = 1, \cdots, N$ and thus is Schur. It easily follows from Theorem 1 that the bipartite formation control is achieved under output feedback controller (50) and convergence of formation tracking error (49) is also guaranteed. ■

IV. NUMERICAL EXAMPLES

To illustrate the effectiveness of the designed control laws via state feedback and output feedback, a discrete-time multi-vehicle system composed of seven vehicles is considered in this section. The directed signed graph is shown in Figure 3, where follower vehicles 1-3 are in one subgroup and follower vehicles 4-6 are in another subgroup. Specially, only vehicles 1 and 4 can get information directly from leader vehicle 0. The corresponding topological matrix $\overline{\mathcal{L}}^s$ is

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

The discrete-time linear dynamics of the follower vehicles can be given as follows [23]:

$$\begin{cases} x_{1i}(t+1) = x_{1i}(t) + x_{2i}(t) \\ x_{2i}(t+1) = a_{wi}x_{2i}(t) + b_{wi}u_i(t) \\ y_i(t) = x_{1i}(t) \end{cases}$$

where $x_{1i}(t) \in \mathbb{R}^2$, $x_{2i}(t) \in \mathbb{R}^2$ and $y_i(t) \in \mathbb{R}^2$ respectively denote the position, the velocity and the the position output of follower i. Let $a_{wi} = 1 + \triangle a_i$ and $b_{wi} = 1 + \triangle b_i$ with $\triangle a_i, \triangle b_i, i = 1, \cdots, 6$ being perturbed values. Select

$$\triangle a_{1,\cdots,6} = (0.1, -0.2, 0.3, 0.4, -0.3, -0.1)$$
$$\triangle b_{1,\cdots,6} = (0.1, 0.2, 0, 0.3, -0.1, -0.2)$$

which means that the multi-vehicle system is heterogeneous.

The leader is generated by

$$\begin{cases} r_1(t+1) = r_1(t) + r_2(t) \\ r_2(t+1) = r_2(t) \\ y_r(t) = r_1(t) \end{cases}$$

The objective is to control the follower vehicles to track the leader and reach a desired bipartite formation. The desired relative positions between followers and leader are given as

$$\delta_1 = (0,1)^T,\ \delta_2 = (-0.5,-1.5)^T,\ \delta_3 = (1,-1)^T$$
$$\delta_4 = (0,1)^T,\ \delta_5 = (-1,-1)^T,\ \delta_6 = (2,0)^T$$

which indicates that the follower vehicles of two antagonistic subgroups can finally severally form two classes of triangles to cover the homodromous/inverse trajectories of leader. Thus, the system matrix S_v of exosystem is written as

$$S_v = \begin{bmatrix} S_r & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1) Bipartite formation controller (16) via state feedback
The observer parameters are designed as

$$\mu_1 = 1/2, \mu_2 = 1, \mu_3 = 1/2,$$
$$\mu_4 = 1, \mu_5 = 1/3, \mu_6 = 1,$$
$$F = (2.0195; 0.5459)\,, H = (0.5, 0.1)$$

The control gain matrices for the state feedback controller (24) are chosen as

$$K_{11} = \begin{pmatrix} -0.6169 & 0 & -1.5909 & 0 \\ 0 & -0.6169 & 0 & -1.5909 \end{pmatrix},$$
$$K_{12} = \begin{pmatrix} -0.8483 & 0 & -2.3125 & 0 \\ 0 & -0.8483 & 0 & -2.3125 \end{pmatrix},$$
$$K_{13} = \begin{pmatrix} -0.5220 & 0 & -1.2692 & 0 \\ 0 & -0.5220 & 0 & -1.2692 \end{pmatrix},$$
$$K_{14} = \begin{pmatrix} -0.4847 & 0 & -1.3929 & 0 \\ 0 & -0.4847 & 0 & -1.3929 \end{pmatrix},$$
$$K_{15} = \begin{pmatrix} -0.9694 & 0 & -2.2143 & 0 \\ 0 & -0.9694 & 0 & -2.2143 \end{pmatrix},$$
$$K_{16} = \begin{pmatrix} -0.7540 & 0 & -1.6111 & 0 \\ 0 & -0.7540 & 0 & -1.6111 \end{pmatrix},$$
$$K_{21} = \begin{pmatrix} 0.6169 & 0 & 1.5000 & 0 \\ 0 & 0.6169 & 0 & 1.5000 \end{pmatrix},$$
$$K_{22} = \begin{pmatrix} 0.8483 & 0 & 2.0625 & 0 \\ 0 & 0.8483 & 0 & 2.0625 \end{pmatrix},$$
$$K_{23} = \begin{pmatrix} 0.5220 & 0 & 1.2692 & 0 \\ 0 & 0.5220 & 0 & 1.2692 \end{pmatrix},$$
$$K_{24} = \begin{pmatrix} 0.4847 & 0 & 1.1786 & 0 \\ 0 & 0.4847 & 0 & 1.1786 \end{pmatrix},$$
$$K_{25} = \begin{pmatrix} 0.9694 & 0 & 2.3571 & 0 \\ 0 & 0.9694 & 0 & 2.3571 \end{pmatrix},$$
$$K_{26} = \begin{pmatrix} 0.7540 & 0 & 1.8333 & 0 \\ 0 & 0.7540 & 0 & 1.8333 \end{pmatrix},$$

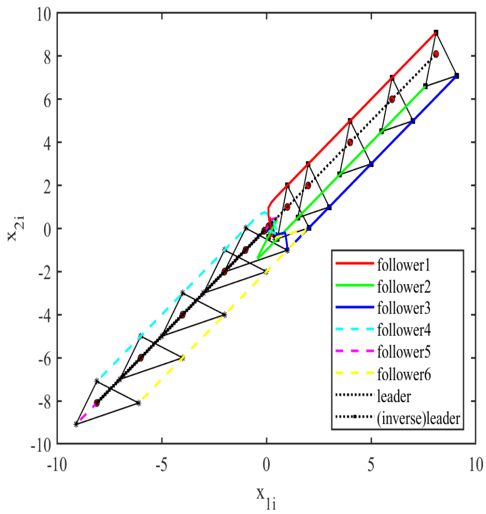

FIGURE 4. Bipartite formation trajectories under distributed controller (16).

$$K_{31} = \begin{pmatrix} 0.6169 & 0 \\ 0 & 0.6169 \end{pmatrix}, K_{32} = \begin{pmatrix} 0.8483 & 0 \\ 0 & 0.8483 \end{pmatrix},$$
$$K_{33} = \begin{pmatrix} 0.5220 & 0 \\ 0 & 0.5220 \end{pmatrix}, K_{34} = \begin{pmatrix} 0.4847 & 0 \\ 0 & 0.4847 \end{pmatrix},$$
$$K_{35} = \begin{pmatrix} 0.9694 & 0 \\ 0 & 0.9694 \end{pmatrix}, K_{36} = \begin{pmatrix} 0.7540 & 0 \\ 0 & 0.7540 \end{pmatrix}$$

(2) Bipartite formation controller (50) via output feedback

Let constant coupling estimator gain $\gamma_{1,\cdots,6} = 1$. The estimator gain matrices are designed as

$$\Gamma_1 = \begin{pmatrix} 1.2100 & 0 & 0.4108 & 0 \\ 0 & 1.2100 & 0 & 0.4108 \end{pmatrix},$$
$$\Gamma_2 = \begin{pmatrix} 1.3100 & 0 & 0.5518 & 0 \\ 0 & 1.3100 & 0 & 0.5518 \end{pmatrix},$$
$$\Gamma_3 = \begin{pmatrix} 1.1100 & 0 & 0.2898 & 0 \\ 0 & 1.1100 & 0 & 0.2898 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} 1.4100 & 0 & 0.7128 & 0 \\ 0 & 1.4100 & 0 & 0.7128 \end{pmatrix},$$
$$\Gamma_5 = \begin{pmatrix} 1.0100 & 0 & 0.1888 & 0 \\ 0 & 1.0100 & 0 & 0.1888 \end{pmatrix},$$
$$\Gamma_6 = \begin{pmatrix} 0.9100 & 0 & 0.1078 & 0 \\ 0 & 0.9100 & 0 & 0.1078 \end{pmatrix}$$

The numerical results under state feedback and output feedback are shown in Figures. 4-7. Figures 4 and 6 respectively depict the bipartite formation trajectories under distributed controllers (16) and (50), where the followers in different subgroups can track the trajectory of exosystem and its negative value according to the given relative positions. Therefore, the bipartite formation composed by two groups of triangulares is ultimately achieved under the proposed control protocols. Figures 5 and 7 show that in both cases all bipartite tracking errors $e_i(t)$ will approach 0 asymptotically, which further illustrate that the proposed distributed bipartite output formation control protocols are effective.

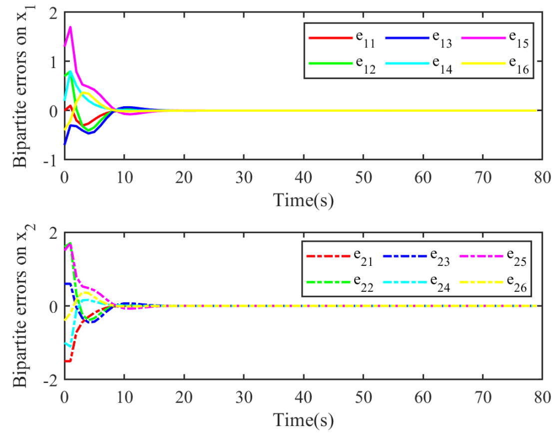

FIGURE 5. Bipartite formation errors under distributed controller (16).

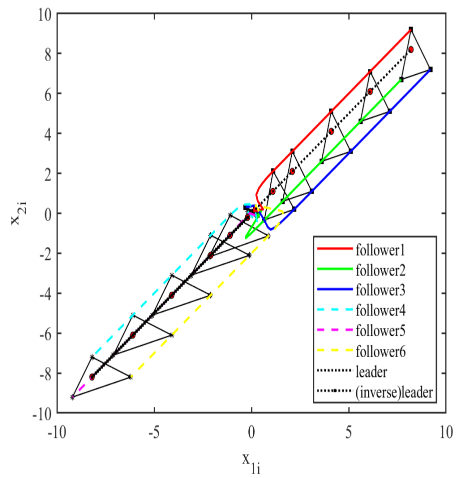

FIGURE 6. Bipartite formation trajectories under distributed controller (50).

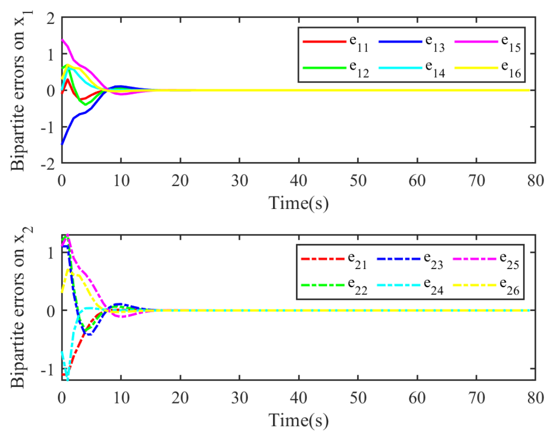

FIGURE 7. Bipartite formation errors under distributed controller (50).

V. CONCLUSION

This paper has presented two classes of distributed bipartite output formation control protocols via state feedback and output feedback under the directed signed topological stricture for heterogenous leader-following MASs. The core concept is to transform the classical consensus control problem into a bipartite formation control problem by considering the output tracking errors as the exogenous signals. Specially, the dynamical systems of followers are discrete-time and non-identical, and the corresponding cooperative control problems can not be easily solved by classical consensus-based distributed control methods. Therefore, a new cooperative output regulation framework associated with bipartite formation has been obtained by introducing the distributed exosystem observer and state estimator in our paper. The numerical simulation results have been provided to show the effective of the designed controllers. In the future work, we will study the bipartite formation-containment problem of heterogenous discrete-time MASs based upon the distributed adaptive exosystem observer.

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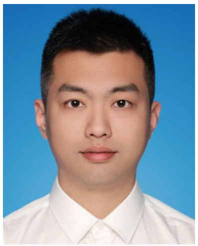

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