A PML-FDTD Algorithm for General Dispersive Media in GPR and Plasma Applications

Guo-Xin Fan and Qing Huo Liu
Klipsch School of Electrical and Computer Engineering
New Mexico State University
Las Cruces, NM 88003, USA

I. Introduction

In recent years, finite-difference time-domain (FDTD) method has been extended to simulate wave propagation in the frequency dispersive media. Most previous work on frequency dependent FDTD method for an unbounded domain uses non-perfectly matched layer (PML) absorbing boundary conditions (ABCs) to truncate the computational domain. In this paper, a 3D FDTD algorithm is presented for general inhomogeneous, dispersive, conductive media. The PML is used as ABC, and the recursive convolution (RC) and piecewise linear recursive convolution (PLRC) approaches [1–2] are extended to general dispersive media in a more unified form. Three types of dispersive media, i.e., Lorentz media, Debye media, and unmagnetized plasma, are treated as special cases of our general formulas. Several ground probing radar (GPR) and plasma application examples are demonstrated.

II. Formulation

Consider an isotropic, conductive, inhomogeneous, linear permittivity dispersive medium. Using the coordinate stretching approach [3] and following a similar procedure as in [4], the modified Maxwell's curl equations with the split fields ($\eta = x, y, z$) in the time domain can be written as

$$\frac{\partial}{\partial \eta}(\hat{\eta} \times \mathbf{E}) = -a_\eta \mu \frac{\partial \mathbf{H}^{(\eta)}}{\partial t} - \mu \omega_\eta \mathbf{H}^{(\eta)} - \mathbf{M}^{(\eta)} \tag{1}$$

$$\frac{\partial}{\partial \eta}(\hat{\eta} \times \mathbf{H}) = a_\eta \frac{\partial \mathbf{D}^{(\eta)}}{\partial t} + \omega_\eta \mathbf{D}^{(\eta)} + a_\eta \sigma \mathbf{E}^{(\eta)} + \omega_\eta \sigma \int_{-\infty}^{t} \mathbf{E}^{(\eta)}\, dt + \mathbf{J}^{(\eta)}. \tag{2}$$

Equations (1) and (2) consist of a total of 12 scalar equations, since both $\mathbf{E}^{(\eta)}$ and $\mathbf{H}^{(\eta)}$ have two scalar components perpendicular to $\hat{\eta}$, and $\mathbf{D}^{(\eta)}$ also has the two corresponding components due to the constitutive relations of the medium. These equations are insufficient to solve the total 18 field components. The remaining equations will be given by the constitutive relations.

For a linear dispersive medium, the relationship between the electric flux density and the electric field intensity in the time domain is described by

$$\mathbf{D}(t) = \epsilon_0 \epsilon_\infty \mathbf{E}(t) + \epsilon_0 \int_{-\infty}^{t} \mathbf{E}(\tau)\, \chi(t-\tau)\, d\tau \tag{3}$$

where ϵ_0 is the free-space permittivity, ϵ_∞ is the relative permittivity at $\omega \to \infty$, and χ is the electric susceptibility.

This work was supported by Environmental Protection Agency through at PECASE grant CR-825-225-010, and by the National Science Foundation through a CAREER grant ECS-9702195.

0-7803-4478-2/98/$10.00 © 1998 IEEE

The frequency domain susceptibility functions, as the transfer function of a linear system, can be generally expressed as a ratio of two polynomials [1] or in a fractional form, i.e.,

$$\chi(\omega) = \sum_{q=1}^{M'} \beta_q s^q \Big/ \sum_{q=1}^{M} \zeta_q s^q = \sum_{q=1}^{M} \frac{\Gamma_q}{s - s_q}, \qquad (M > M') \tag{4}$$

where $s = -i\omega$, and s_q and Γ_q are the complex poles and the corresponding residues. Then the corresponding time domain susceptibility functions can be written as

$$\chi(t) = \sum_{q=1}^{N} \mathrm{Re}[R_q e^{s_q t} U(t)] \tag{5}$$

where $U(t)$ is the unit step function. In (5), $N = M$, $R_q = \Gamma_q$ when all s_q and Γ_q are real; and $N = M/2$, $R_q = 2\Gamma_q$ when there are $M/2$ complex-conjugate pole pairs (such as Lorentz media) which satisfy $\Gamma_q(s_q) = \Gamma_q^*(s_q^*)$ since $\chi(t)$ is a real function.

To simplify (3), we first introduce a unified piecewise approximation to $\mathbf{E}(t)$ over the time interval $t \in [m\Delta t, (m+1)\Delta t]$ as follows,

$$\mathbf{E}(t) \approx \mathbf{E}(m+1) + K_a \frac{[\mathbf{E}(m+1) - \mathbf{E}(m)]}{\Delta t} [t - (m+1)\Delta t]. \tag{6}$$

It is noted that Equation (6) corresponds to the recursive convolution (RC) [1] when $K_a = 0$, and to the piecewise linear recursive convolution (PLRC) [2] when $K_a = 1$.

Using (5) and the unified approximation (6), the convolution integral in (3) is then transformed into the discrete convolution summation,

$$\mathbf{D}(n) = \epsilon_0 \epsilon_\infty \mathbf{E}(n) + \epsilon_0 \sum_{q=1}^{N} \mathrm{Re}[\Psi_q(n)] \tag{7}$$

where

$$\Psi_q(n) = \sum_{m=0}^{n-1} \left\{ \left[\hat{\chi}_q(0) - \hat{\xi}_q(0) \right] \mathbf{E}(n-m) + \hat{\xi}_q(0) \mathbf{E}(n-m-1) \right\} e^{s_q m \Delta t} \tag{8}$$

$$\hat{\chi}_q(0) = \begin{cases} R_q \Delta t; \\ \dfrac{R_q}{s_q}(e^{s_q \Delta t} - 1), \end{cases} \quad \hat{\xi}_q(0) = \begin{cases} \dfrac{K_a R_q \Delta t}{2}; & \text{for } s_q = 0; \\ \dfrac{K_a R_q}{\Delta t s_q^2} \left[1 - (1 - s_q \Delta t) e^{s_q \Delta t} \right], & \text{for } s_q \neq 0. \end{cases} \tag{9}$$

From the above formulas, we derived the recursive relations of $\Psi_q(n+1)$, and $\mathbf{D}(n+1)$. With the help of these recursive convolution equations, we can proceed to solve Maxwell's equations by using the Yee's algorithm to discretize the split equations (1) and (2). We obtain the electric field time-stepping equations as follows

$$c_1^{(\eta)} \mathbf{E}^{(\eta)}(n+1) = \frac{\partial}{\partial \eta} \left[\hat{\eta} \times \mathbf{H}(n + \tfrac{1}{2}) \right] + \Phi^{(\eta)}(n) + c_0^{(\eta)} \mathbf{E}^{(\eta)}(n) - \sigma \omega_\eta \Delta t \mathbf{E}_I^{(\eta)}(n) - \mathbf{J}^{(\eta)}(n + \tfrac{1}{2}) \tag{10}$$

where $c_0^{(\eta)}$ and $c_1^{(\eta)}$ are the known coefficients, and

$$\Phi^{(\eta)}(n) = \epsilon_0 \sum_{q=1}^{N} \mathrm{Re}\left\{\left[\left(\frac{a_\eta}{\Delta t} - \frac{\omega_\eta}{2}\right) - \left(\frac{a_\eta}{\Delta t} + \frac{\omega_\eta}{2}\right) e^{-s_q \Delta t}\right] \Psi_q^{(\eta)}(n)\right\} \tag{11}$$

$$\mathbf{E}_I^{(\eta)}(n) = \mathbf{E}_I^{(\eta)}(n-1) + \frac{7}{8}\mathbf{E}^{(\eta)}(n) + \frac{1}{8}\mathbf{E}^{(\eta)}(n-1). \tag{12}$$

The magnetic field time-stepping equation is unchanged for magnetically non-dispersive media, as given in [4].

Up to now, we give the algorithm for general dispersive media. For a given medium, we only need a corresponding set of R_q and s_q in advance. Here, we also point out that for an arbitrary linear dispersive medium, when the discrete spectral magnitude data are available for the susceptibility of the medium, the frequency-domain Prony method (FDPM) can be used to find directly the poles and residues, i.e. R_q and s_q [5]. Therefore, for an arbitrary dispersive medium there is no need to fit the dispersive relation with Debye or Lorentz models.

III. Numerical Results

The algorithm and computer program has first been applied to non-dispersive media by letting the electric susceptibility or the corresponding parameters go to zero. The results obtained are compared with those of [4], and excellent agreement is observed in various cases. We also validated the algorithm for both homogeneous dispersive media and a dispersive sphere in another dispersive or non-dispersive background medium for three typical kinds of dispersive media, i.e., Lorentz medium, unmagnetized plasma, and Debye medium. Excellent agreement between the FDTD results and analytical solutions is obtained for all cases.

To demonstrate the effectiveness of the algorithm, we consider several applications of GPR and plasma. In following examples, an electric dipole directed in $\hat{x}$ direction is used as a source, and time function of the source is the first derivative of the Blackman-Harris window function [4]. Fig. 1 shows the scattered field from two buried mine-like objects. The buried objects are non-dispersive. The electrical parameters of the earth are chosen as Debye media. Figs. 2(c) and 2(b) show the transient radiation from a waveguide backed slot covered with and without the plasma, respectively.

IV. Conclusions

We present a 3D FDTD algorithm with the PML absorbing boundary condition for general inhomogeneous, dispersive, conductive media. The modified time-domain Maxwell's equations for dispersive media are expressed in terms of the coordinate-stretching variables. A single formulation is developed to include recursive convolution and piecewise linear recursive convolution for arbitrary dispersive media. Several applications are demonstrated for subsurface radar detection of cylinders and a sphere buried in a dispersive half-space. The algorithm proposed is ideal for parallel computation since the same code is shared by both the interior computational region and the outer matched layers. Because of their generality, the algorithm and computer program developed can be used to model biological materials, artificial dielectrics, optical materials, and other dispersive media.

References

[1] R. J. Luebbers, D. Steich, and K. Kunz, "FDTD calculation of scattering from frequency-dependent materials," *IEEE Trans. Antennas Propagat.*, vol. 41, pp. 1249-1257, 1993.

[2] D. F. Kelley and R. J. Luebbers, "Piecewise linear recursive convolution for dispersive media using FDTD," *IEEE Trans. Antennas Propagat.*, vol. 44, pp. 792-797, 1996.
[3] W. C. Chew and W. H. Weedon, "A 3D perfectly matched medium from modified Maxwell's equation with stretched coordinates," *Microwave Opt. Tech. Lett.*, vol. 7, pp. 599-604, 1994.
[4] Q. H. Liu, "An FDTD algorithm with perfectly matched layers for conductive media," *Microwave Opt. Tech. Lett.*, vol. 14, pp. 134-137, 1997.
[5] J. N. Brittingham, E.K. Miller, and J. L. Wilows, "Pole extraction from real-frequency information," *Proc. IEEE*, vol. 68, pp. 263-273, 1980.

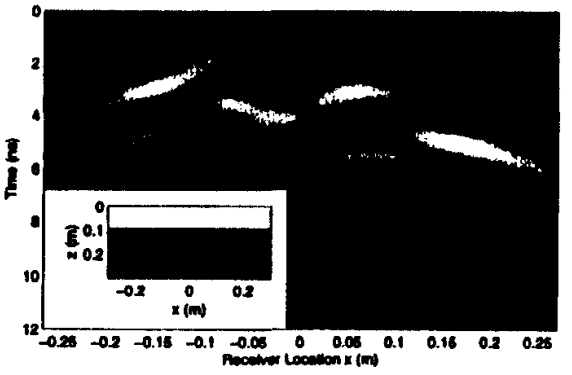

Figure 1. The E_x field distribution of two mine-like disks buried in a Debye medium half-space.

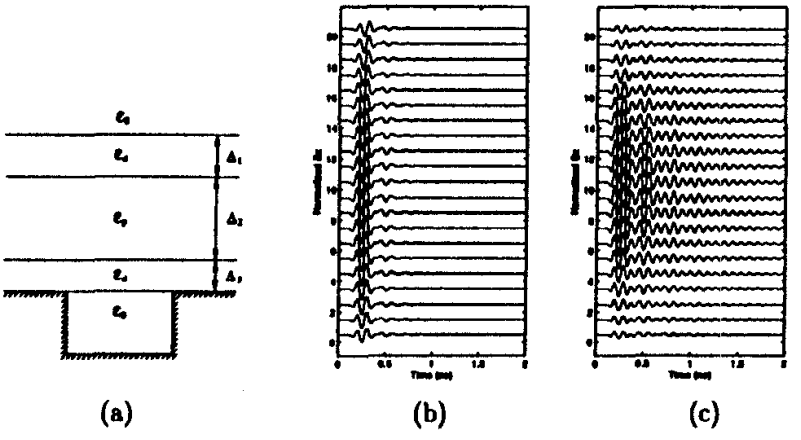

Figure 2. The transient radiated field from a plasma-covered waveguide backed slot. (a) Geometry; The E_x field distribution without (b) and with (c) plasma.