Dyadic Green's Functions for Curved Waveguides and Cavities and Their Reformulation

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I. Introduction

Continuously curved waveguides are used not only for waveguide bends but also for feeding or radiating guides for cylindrical conformal slot arrays [1]. In solving the curved waveguide junction problems and the coupling slot and radiating slot problems, one often uses the dyadic Green's functions (DGFs) for curved waveguides and cavities. Although many expressions of DGFs have been derived for various straight waveguides and cavities, no expressions of DGFs for curved waveguides and cavities are available. The difficulty is in that there only exist the longitudinal-section electric modes (LSE) and longitudinal-section magnetic modes (LSM) in curved waveguides. But these two kinds of modes do not have the complete orthogonality in the terms of the dot product. Therefore, the usual Ohm-Rayleigh method is not applicable to the curved waveguides problems. Moreover, unlike common rectangular waveguides, the complete expression of DGFs for curved guides cannot be constructed from the potential DGFs. On the other hand, when applied to boundary integral equations, the usual expressions of the DGFs is not convenient for numerical computation because of their higher singularity and discontinuity. In this paper, the DGFs for curved waveguides and cavities are constructed by the fields excited by a unit electric or magnetic source. The field expressions are derived with the help of the Lorentz reciprocity theorem and the mode orthogonality, and by adding the source-point term. The common form of DGFs is reformulated into a form convenient for numerical computation. Finally, a general procedure for reformulation of DGFs for waveguides is proposed.

II. Vector Wave Functions in Curved Waveguides

Consider a curved waveguide (or cavity) of rectangular cross section with perfectly conducting walls. With respect to a cylindrical coordinate system (ρ, φ, z), the inner waveguide walls are defined by $\rho = \rho_1, \rho_2$ and $z = 0, c$, and the two ends by $\varphi = \varphi_1, \varphi_2$. There are four possible cases for ends conditions: both ends are matched (Case I) or shorted (Case II), the end $\varphi = \varphi_1$ is matched while the end $\varphi = \varphi_2$ is shorted (Case III), or vise versa (Case IV).

The eigenmodes in curved waveguides can be classified as either LSE modes or LSM modes. For the LSE modes, the corresponding vector wave functions are defined as

$$\mathbf{M}^{\pm}_{\mu mn} = \nabla \times \hat{z}\psi^{\pm}_{\mu mn}, \qquad \mathbf{N}^{\pm}_{\mu mn} = (1/k)\nabla \times \nabla \times \hat{z}\psi^{\pm}_{\mu mn} \tag{1}$$

$$\psi^{\pm}_{\mu mn} = B_{\mu_{mn}}(h_m\rho)\sin(\frac{m\pi z}{c})\Phi^{\pm}_{\mu_{mn}}(\varphi) \tag{2}$$

$$B_{\mu_{mn}}(h_m\rho) = J'_{\mu_{mn}}(h_m\rho_1)Y_{\mu_{mn}}(h_m\rho) - Y'_{\mu_{mn}}(h_m\rho_1)J_{\mu_{mn}}(h_m\rho) \tag{3}$$

$$\Phi^{\pm}_{\mu_{mn}}(\varphi) = \begin{cases} \begin{cases} \exp(+j\mu_{mn}\varphi) \\ \exp(-j\mu_{mn}\varphi) \end{cases} \Big/ \begin{cases} \sin\mu_{mn}(\varphi-\varphi_1) \\ \sin\mu_{mn}(\varphi_2-\varphi) \end{cases} & \text{for Case I / Case II} \\ \begin{cases} \exp(+j\mu_{mn}\varphi) \\ \sin\mu_{mn}(\varphi_2-\varphi) \end{cases} \Big/ \begin{cases} \sin\mu_{mn}(\varphi-\varphi_1) \\ \exp(-j\mu_{mn}\varphi) \end{cases} & \text{for Case III / Case IV} \end{cases} \tag{4}$$

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where k is the wavenumber, $h_m = \sqrt{k^2 - (m\pi/c)^2}$ ($\Im m(h_m) \leq 0$), J and Y are Bessel functions of first and second kind, the prime on the functions denotes derivative with respect to the argument, the eigenvalues μ_{mn} are the roots of the equation $B'_{\mu_{mn}}(h_m\rho_2) = 0$.

For the LSM modes, the corresponding vector wave functions are defined as

$$\mathbf{M}^{\pm}_{\varepsilon mn} = \nabla \times \hat{z}\psi^{\pm}_{\varepsilon mn}, \qquad \mathbf{N}^{\pm}_{\varepsilon mn} = (1/k)\nabla \times \nabla \times \hat{z}\psi^{\pm}_{\varepsilon mn} \tag{5}$$

$$\psi^{\pm}_{\varepsilon mn} = C_{\varepsilon_{mn}}(h_m\rho)\cos(\frac{m\pi z}{c})\Phi^{\pm}_{\varepsilon_{mn}}(\varphi) \tag{6}$$

$$C_{\varepsilon_{mn}}(h_m\rho) = J_{\varepsilon_{mn}}(h_m\rho_1)Y_{\varepsilon_{mn}}(h_m\rho) - Y_{\varepsilon_{mn}}(h_m\rho_1)J_{\varepsilon_{mn}}(h_m\rho) \tag{7}$$

where the eigenvalues are the roots of the equation $C_{\varepsilon_{mn}}(h_m\rho_2) = 0$, and $\Phi^{\pm}_{\varepsilon_{mn}}(\varphi)$ takes the same forms of $\Phi^{\pm}_{\mu_{mn}}(\varphi)$ except for μ_{mn} replaced by ε_{mn} and sine-function by cosine-function.

It can be shown the the vector wave functions above have the following orthogonal relations:

$$\int_{\rho_1}^{\rho_2}\int_0^c (\mathbf{M}^{\pm}_{\mu mn} \times \mathbf{N}^{\pm}_{\varepsilon m'n'}) \cdot \hat{\varphi}\, d\rho\, dz = \int_{\rho_1}^{\rho_2}\int_0^c (\mathbf{N}^{\pm}_{\mu mn} \times \mathbf{M}^{\pm}_{\varepsilon m'n'}) \cdot \hat{\varphi}\, d\rho\, dz = 0 \tag{8}$$

$$\int_{\rho_1}^{\rho_2}\int_0^c (\mathbf{M}^{\pm}_{\mu mn} \times \mathbf{N}^{\mp}_{\mu m'n'}) \cdot \hat{\varphi}\, d\rho\, dz = -\frac{1+\delta_{0m}}{2}\frac{h_m^2 c}{k}\delta_{mm'}\delta_{nn'}N_{\mu_{mn}}\Phi^{\pm'}_{\mu_{mn}}(\varphi)\Phi^{\mp}_{\mu_{mn}}(\varphi) \tag{9}$$

$$\int_{\rho_1}^{\rho_2}\int_0^c (\mathbf{M}^{\pm}_{\varepsilon mn} \times \mathbf{N}^{\mp}_{\varepsilon m'n'}) \cdot \hat{\varphi}\, d\rho\, dz = -\frac{1+\delta_{0m}}{2}\frac{h_m^2 c}{k}\delta_{mm'}\delta_{nn'}N_{\varepsilon_{mn}}\Phi^{\pm'}_{\varepsilon_{mn}}(\varphi)\Phi^{\mp}_{\varepsilon_{mn}}(\varphi) \tag{10}$$

where $N_{\mu_{mn}}$ and $N_{\varepsilon_{mn}}$ are the mode normalized constants. These vector wave functions also form a complete set in the source-free region, and the fields outside the source region can be uniquely expanded in terms of these functions. Note that the above orthogonality is based on the concept of power flow not on those of dot-product as required in Ohm-Rayleigh method.

III. Magnetic-Source Dyadic Green's Functions

The magnetic-source electric-field DGF $\overline{\mathbf{G}}^{(m)}_E(\mathbf{r},\mathbf{r}')$ and magnetic-source magnetic-field DGF $\overline{\mathbf{G}}^{(m)}_H(\mathbf{r},\mathbf{r}')$ satisfy the following differential equations and boundary conditions, respectively

$$\nabla \times \nabla \times \overline{\mathbf{G}}^{(m)}_E - k^2\overline{\mathbf{G}}^{(m)}_E = -\nabla \times \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'), \qquad \hat{n} \times \overline{\mathbf{G}}^{(m)}_E = 0 \quad \text{on the walls} \tag{11}$$

$$\nabla \times \nabla \times \overline{\mathbf{G}}^{(m)}_H - k^2\overline{\mathbf{G}}^{(m)}_H = -j\omega\epsilon\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'), \qquad \hat{n} \times \nabla \times \overline{\mathbf{G}}^{(m)}_H = 0 \quad \text{on the walls} \tag{12}$$

The solutions of (11) and (12) can be written in the form [2]

$$\overline{\mathbf{G}}^{(m)}_E(\mathbf{r},\mathbf{r}') = \sum_j \mathbf{E}^{(m)}(\mathbf{r},\mathbf{r}';\hat{u}'_j)\hat{u}'_j, \qquad \overline{\mathbf{G}}^{(m)}_H(\mathbf{r},\mathbf{r}') = \sum_j \mathbf{H}^{(m)}(\mathbf{r},\mathbf{r}';\hat{u}'_j)\hat{u}'_j \tag{13}$$

where $\hat{u}'_j (j = 1, 2, 3)$ is a unit vector in an orthogonal coordinate system, and $\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'; \hat{u}'_j)$ and $\mathbf{H}^{(m)}(\mathbf{r}, \mathbf{r}'; \hat{u}'_j)$ are the electric and magnetic fields at the field point $\mathbf{r}$ excited by a unit point magnetic dipole in the $\hat{u}'_j$-direction at the source point $\mathbf{r}'$.

The solution procedure of $\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'; \hat{u}'_j)$ and $\mathbf{H}^{(m)}(\mathbf{r}, \mathbf{r}'; \hat{u}'_j)$ is divided in two steps. First, outside the the source region, these fields are expanded in terms of the eigenvector functions, $\mathbf{M}_{\epsilon mn}$, $\mathbf{N}_{\epsilon mn}$, $\mathbf{M}_{\mu mn}$ and $\mathbf{N}_{\mu mn}$, and the expansion coefficients are determined by using the Lorentz reciprocity and the orthogonal relations given in the previous section. Then, the source-point term is obtained by using the result of [3] and the duality principle or by the procedure as introduced in [2]. Finally, we get

$$\overline{\mathbf{G}}_E^{(m)}(\mathbf{r}, \mathbf{r}') = \sum_m \sum_n \left[a_{mn}\mathbf{N}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{M}^{\mp}_{\epsilon mn}(\mathbf{r}') + b_{mn}\mathbf{M}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{N}^{\mp}_{\epsilon mn}(\mathbf{r}')\right] \quad \varphi \gtrless \varphi' \quad (14)$$

$$\overline{\mathbf{G}}_H^{(m)}(\mathbf{r}, \mathbf{r}') = \frac{-k}{j\omega\mu} \sum_m \sum_n \left[a_{mn}\mathbf{M}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{M}^{\mp}_{\epsilon mn}(\mathbf{r}') + b_{mn}\mathbf{N}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{N}^{\mp}_{\epsilon mn}(\mathbf{r}')\right]$$
$$+ \hat{\varphi}\hat{\varphi}'\delta(\mathbf{r} - \mathbf{r}')/(-j\omega\mu) \qquad \varphi \gtrless \varphi' \quad (15)$$

$$a_{mn} = -\frac{2}{1 + \delta_0 m} \frac{k}{h_m^2 c N_{\epsilon_{mn}} \Delta_{\epsilon_{mn}}}, \qquad b_{mn} = -\frac{2}{1 + \delta_0 m} \frac{k}{h_m^2 c N_{\mu_{mn}} \Delta_{\mu_{mn}}} \quad (16)$$

where $\Delta_{\mu_{mn}}$ and $\Delta_{\epsilon_{mn}}$ are the constants dependent on end condition (for example, $\Delta_{\mu_{mn}} = 2j\mu_{mn}$ and $\Delta_{\epsilon_{mn}} = 2j\epsilon_{mn}$ for Case I).

The electric-source electric- and magnetic-field DGFs are also obtained by using the above procedure or the duality principle.

III. Reformulation of Dyadic Green's Functions

When the DGFs derived above are applied to the boundary integral equation, there exist two main computational difficulties if the source and field points coincide: one is the treatment of δ-function, the other is their higher singularity. In this section, we derive a new form of the DGF on the curved walls of waveguide.

Because of the localized effect of δ-function, the unit vector $\hat{p}$ in the source-point term $\hat{p}\hat{p}'\delta(\mathbf{r} - \mathbf{r}')$ for an given $\hat{p}$-propagating direction can be considered as a constant vector. By introducing the eigenvector functions of TE and TM modes to $\hat{p}$-direction, $\mathbf{M}_\epsilon^{(E)}$, $\mathbf{N}_\epsilon^{(E)}$, $\mathbf{M}_\mu^{(H)}$, $\mathbf{N}_\mu^{(H)}$, and defining $\mathbf{L}_\epsilon^{(E)} = \nabla\psi_\epsilon^{(E)}$ and $\mathbf{L}_\mu^{(H)} = \nabla\psi_\mu^{(H)}$, we can expand the δ-function in the source-point terms for magnetic field and electric field, respectively, as follows,

$$\delta_H(\mathbf{r} - \mathbf{r}') = \sum_\mu A_\mu \frac{1}{g_p} \frac{\partial\psi_\mu^{(H)}}{\partial p}, \qquad \delta_E(\mathbf{r} - \mathbf{r}') = \sum_\epsilon B_\epsilon \frac{1}{g_p} \frac{\partial\psi_\epsilon^{(E)}}{\partial p} \quad (17)$$

where g_p is the metric coefficient. For curved waveguides, it is shown that

$$\hat{\varphi}\hat{\varphi}'\delta(\mathbf{r} - \mathbf{r}') = \hat{\varphi}\hat{\varphi}'\delta(\varphi - \varphi') \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{2}{1 + \delta_{0m}} \frac{B_{\mu_{mn}}(h_m\rho) B_{\mu_{mn}}(h_m\rho')}{c N_{\mu_{mn}} \rho\rho'} C_z C_{z'} \quad (18)$$

where $C_z = \cos(m\pi z/c)$. Using the above result, from the expression of the magnetic-source magnetic-field DGF, after a lengthily manipulation, we obtain

$$\overline{\mathbf{G}}_H^{(m)} = \frac{k^2}{j\omega\mu} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{[C'_{\epsilon_{mn}}(h_m\rho_2)]^2}{(1 + \delta_{0m}) N_{\epsilon_{mn}} a\, j\epsilon_{mn}} \hat{\varphi}\hat{\varphi}' C_z C_{z'} e^{-j\epsilon_{mn}|\varphi - \varphi'|}$$

$$+\frac{k^2}{j\omega\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[B_{\mu_{mn}}(h_m\rho_2)]^2}{(1+\delta_{0m})N_{\mu_{mn}}a\,j\mu_{mn}}\left\{\left[\hat{z}\hat{z}'S_zS_{z'}+\hat{\varphi}\hat{\varphi}'\frac{\mu_{mn}^2}{h_m^2\rho_2^2}C_zC_{z'}\right]e^{-j\mu_{mn}|\varphi-\varphi'|}\right.$$
$$\left.-\frac{1}{k^2}\nabla_t\nabla_t'C_zC_{z'}e^{-j\mu_{mn}|\varphi-\varphi'|}\right\} \tag{19}$$

where $a = c$, $S_z = \sin(m\pi z/c)$, and ∇_t is the tangential derivatives at the surface $\rho = \rho_2$ where the source and field points are located. Note that in the new expression, the δ-function term disappears, and the operator ∇_t is introduced. It is the operators $\nabla_t\nabla_t'$ that increase the order of singularity associated with the dyadic Green's functions. The form $\nabla_t\nabla_t'$ is convenient for the follow-up treatment when applying the moment method to surface integral equations.

A heuristic procedure can also be used for the reformulation of DGFs. We first construct the operator $\nabla_t\nabla_t'$ form, then substitute it into the original expression of DGF, and get the new form via an arrangement process.

This heuristic procedure can be generalized to the reformulation of the DGFs for a general waveguide. For an orthogonal curvilinear coordinate system (u, v, p), we define $\hat{p}$ as the propagating direction, and $u = u' = u_0$ (at the wall). We give a general reformulation procedure as follows,

1) Expand the $\delta(\mathbf{r}-\mathbf{r}')$ function term using (17) according to the type of the DGF;
2) Introduce the tangential derivatives at the coordinate plane where the waveguide wall is located
$$\nabla_t = \hat{p}\frac{1}{g_p}\frac{\partial}{\partial p}+\hat{v}\frac{1}{g_v}\frac{\partial}{\partial v}, \qquad \nabla_t' = \hat{p}'\frac{1}{g_{p'}}\frac{\partial}{\partial p'}+\hat{v}'\frac{1}{g_{v'}}\frac{\partial}{\partial v'} \tag{20}$$
and calculate the term $(-1/k^2)\nabla_t\nabla_t' g_{mn}(\mathbf{r}-\mathbf{r}')$, where $g_{mn}(\mathbf{r}-\mathbf{r}')$ is the expansion function of the $\hat{p}\hat{p}'$-component of the DGF and takes the same form as the eigen function to expand the $\delta(\mathbf{r}-\mathbf{r}')$ function;
3) Substitute the above result into the original expression of the DGF, cancel the source-point term, and rearrange the terms in the expression into the desired form with the vector operator $(-1/k^2)\nabla_t\nabla_t'$.

The procedure has been used for the transformation of the DGFs for straight waveguides with rectangular and sectorial cross sections.

IV. Conclusions

The DGFs for curved waveguides and cavities are derived through a simple procedure, and reformulated into a new form convenient for numerical computation when both the source and field points are located at the same waveguide wall. A general procedure for reformulation of DGFs is proposed for general waveguides. The DGFs derived can be used for solving the problems with curved waveguide coupling and radiating slots, as well as waveguide junction problems.

References

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