A Nonuniform Cylindrical FDTD Algorithm Using New Split PML Formulations

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I. Introduction

The perfectly matched layers (PML) was first introduced by Berenger as a material absorbing boundary conditions (ABC) for electromagnetic waves [1]. It has been developed widely for simulation of wave propagation and scattering in an unbounded medium in Cartesian coordinates. Rigorous studies have been done on the PML for nonorthogonal grids, notably [2-4]. Most schemes do not admit cylindrical harmonics as the eigensolutions of the modified Maxwell's equations. Only recently, several formulations of PML ABCs have been implemented in cylindrical coordinates [5-9]. In this work we show a unified formulation to develop two PML ABCs which split Maxwell's equations in a form suitable for time-stepping. A nonuniform grid is used to implement FDTD algorithm for inhomogeneous conductive media in order to save computer memory and computational time. Numerical results agree well with analytical solutions. Applications of the 3-D algorithm are shown for borehole radar probing.

II. Formulations

For simplicity, we first consider Maxwell's equations in two-dimensional polar coordinates for the TE_z case. Here we adopt the concept of complex coordinates in [7] to present two PML formulations for polar coordinates. We use the complex coordinate stretching variables e_r and e_θ such that $\frac{\partial}{\partial r} \to \frac{1}{e_r}\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta} \to \frac{1}{e_\theta}\frac{\partial}{\partial \theta}$. In general, $e_r = a_r + i\omega_r/\omega$, while e_θ is different for the two formulations. Then, Maxwell's equations in polar coordinates are modified as

$$i\omega\epsilon E_r = -\frac{1}{re_\theta}\frac{\partial H_z}{\partial \theta}, \tag{1a}$$

$$i\omega\epsilon E_\theta = \frac{1}{e_r}\frac{\partial H_z}{\partial r}, \tag{1b}$$

$$-i\omega\mu H_z = -\frac{1}{\tilde{r}e_r}\frac{\partial(\tilde{r}E_\theta)}{\partial r} + \frac{1}{re_\theta}\frac{\partial E_r}{\partial \theta}, \tag{1c}$$

where $\tilde{r} = \tilde{r}(r)$ is in general a complex function which distinguishes the quasi-PML and PML formulations.

(a) PML scheme using complex coordinates (CPML)

If we choose $e_\theta = \tilde{r}/r$, $e_r = a_r(r) + i\omega_r(r)/\omega$, and the complex radial coordinate

$$\tilde{r} = \int_0^r e_r(r')dr' = \int_0^r \left[a_r(r') + i\frac{\omega_r(r')}{\omega}\right]dr' = A_r(r) + i\frac{\Omega_r(r)}{\omega}, \tag{2}$$

and rewrite (1c) as

$$i\omega\mu(H_z^{(r)} + H_z^{(\theta)}) = \frac{E_\theta}{\tilde{r}e_r}\frac{\partial \tilde{r}}{\partial r} + \frac{1}{e_r}\frac{\partial E_\theta}{\partial r} - \frac{1}{re_\theta}\frac{\partial E_r}{\partial \theta}, \tag{3}$$

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equations (1a)-(1c), after splitting $H_z = H_z^{(r)} + H_z^{(\theta)}$, can be cast in the following form:

$$i\omega\tilde{r}\epsilon E_r = -\frac{\partial(H_z^{(r)} + H_z^{(\theta)})}{\partial\theta}, \tag{4a}$$

$$i\omega e_r \epsilon E_\theta = \frac{\partial(H_z^{(r)} + H_z^{(\theta)})}{\partial r}, \tag{4b}$$

$$i\omega e_r \mu H_z^{(r)} = \frac{\partial E_\theta}{\partial r}, \tag{4c}$$

$$i\omega\tilde{r}\mu H_z^{(\theta)} = E_\theta - \frac{\partial E_r}{\partial\theta}. \tag{4d}$$

Equations 4(a)-(d) can be easily transformed to time domain.

Based on this scheme, the set of time-domain equations for updating **E** field for conductive media in 3-D cylindrical coordinates are

$$A_r\epsilon\frac{\partial E_r^{(\theta)}}{\partial t} + (\Omega_r\epsilon + A_r\sigma)E_r^{(\theta)} + \Omega_r\sigma\int_{-\infty}^{t} E_r^{(\theta)}(\tau)d\tau = \frac{\partial H_z}{\partial\theta} - J_r^{(\theta)}, \tag{5a}$$

$$a_z\epsilon\frac{\partial E_r^{(z)}}{\partial t} + (\omega_z\epsilon + a_z\sigma)E_r^{(z)} + \omega_z\sigma\int_{-\infty}^{t} E_r^{(z)}(\tau)d\tau = -\frac{\partial H_\theta}{\partial z} - J_r^{(z)}, \tag{5b}$$

$$a_r\epsilon\frac{\partial E_\theta^{(r)}}{\partial t} + (\omega_r\epsilon + a_r\sigma)E_\theta^{(r)} + \omega_r\sigma\int_{-\infty}^{t} E_\theta^{(r)}(\tau)d\tau = -\frac{\partial H_z}{\partial r} - J_\theta^{(r)}, \tag{5c}$$

$$a_z\epsilon\frac{\partial E_\theta^{(z)}}{\partial t} + (\omega_z\epsilon + a_z\sigma)E_\theta^{(z)} + \omega_z\sigma\int_{-\infty}^{t} E_\theta^{(z)}(\tau)d\tau = \frac{\partial H_r}{\partial z} - J_\theta^{(z)}, \tag{5d}$$

$$a_r\epsilon\frac{\partial E_z^{(r)}}{\partial t} + (\omega_r\epsilon + a_r\sigma)E_z^{(r)} + \omega_r\sigma\int_{-\infty}^{t} E_z^{(r)}(\tau)d\tau = \frac{\partial H_\theta}{\partial r} - J_z^{(r)}, \tag{5e}$$

$$A_r\epsilon\frac{\partial E_z^{(\theta)}}{\partial t} + (\Omega_r\epsilon + A_r\sigma)E_z^{(\theta)} + \Omega_r\sigma\int_{-\infty}^{t} E_z^{(\theta)}(\tau)d\tau = H_\theta - \frac{\partial H_r}{\partial\theta} - J_z^{(\theta)}. \tag{5f}$$

The other set of equations for updating **H** can be obtained by duality. The set of equations (5) are appropriate for time-stepping. Compared with previous formulation [8], this system requires 12 instead of 14 equations. Compared to [9], it provides additional degree of freedom for the PML of a_r and A_r.

(b) Quasi-PML scheme (QPML)

Note that in the unified formula (1a)-(1c), if we choose $e_\theta = e_r = a_r(r) + i\omega_r(r)/\omega$, and $\tilde{r}(r) = r$, we can rewrite the 3-D updating equations for **E** field in time-domain for conductive media as:

$$a_r\epsilon\frac{\partial E_r^{(\theta)}}{\partial t} + (\omega_r\epsilon + a_r\sigma)E_r^{(\theta)} + \omega_r\sigma\int_{-\infty}^{t} E_r^{(\theta)}(\tau)d\tau = \frac{1}{r}\frac{\partial H_z}{\partial\theta} - J_r^{(\theta)}, \tag{6a}$$

$$a_z\epsilon\frac{\partial E_r^{(z)}}{\partial t} + (\omega_z\epsilon + a_z\sigma)E_r^{(z)} + \omega_z\sigma\int_{-\infty}^{t} E_r^{(z)}(\tau)d\tau = -\frac{\partial H_\theta}{\partial z} - J_r^{(z)}, \tag{6b}$$

$$a_r\epsilon\frac{\partial E_\theta^{(r)}}{\partial t} + (\omega_r\epsilon + a_r\sigma)E_\theta^{(r)} + \omega_r\sigma\int_{-\infty}^{t} E_\theta^{(r)}(\tau)d\tau = -\frac{\partial H_z}{\partial r} - J_\theta^{(r)}, \tag{6c}$$

$$a_z\epsilon\frac{\partial E_\theta^{(z)}}{\partial t} + (\omega_z\epsilon + a_z\sigma)E_\theta^{(z)} + \omega_z\sigma\int_{-\infty}^{t} E_\theta^{(z)}(\tau)d\tau = \frac{\partial H_r}{\partial z} - J_\theta^{(z)}, \tag{6d}$$

$$a_r\epsilon\frac{\partial E_z}{\partial t} + (\omega_r\epsilon + a_r\sigma)E_z + \omega_r\sigma\int_{-\infty}^{t} E_z(\tau)d\tau = \frac{1}{r}\frac{\partial(rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial\theta} - J_z. \tag{6e}$$

Therefore, there are only 10 equations in the 3-D QPML formulations. Although it can be shown theoretically that this PML is not perfectly matched (thus the name quasi-PML) [5], practically it provides a satisfactory ABC.

III. Numerical Results

We use 10 layers of PML in z and ρ directions. The first case is to model a conductive sphere with $\epsilon_r = 4.0$, $\mu_r = 1.0$ and $\sigma = 1\times10^{-4}$ S/m in a nonconductive background medium with $\epsilon_r = 1.0$, $\mu_r = 1.0$ and $\sigma = 0.0$ S/m. A z-directed magnetic dipole ring source is located at the center of the sphere. Figure 1 shows the excellent agreement of the QPML and CPML results with the analytical solution. In the second case, we simulate a line source in free space with the derivative of a Blackman-Harris window time function at a center frequency $f_c = 300$ MHz. Fifteen receivers are set uniformly around a semi-circle. In Figure 2, we show the excellent agreement between QPML,CPML schemes and analytical solution. Figure 3(a), (b) show the geometry of the problem in xz cross section of a 3-D borehole radar detection of vertical and horizontal fractures, and the xy cross section. The magnified cross section in Figure 3(c) shows the grid around the vertical fracture. To calculate the scattered field of the two fractures, using nonuniform FDTD can save about 45 times CPU and memory compared to a uniform one. The scattered field is shown in Figure 4.

IV. Conclusion

Based on a unified formulation, a new split formulation of PML and Quasi-PML are presented. Numerical results of both schemes agree well with analytical solutions, although the quasi-PML can be shown theoretically not perfectly matched. A nonuniform grid in cylindrical coordinates is used for 3-D FDTD method. The new split formulation of PML is an important ABC for spectral-domain methods such as pseudospectral time domain (PSTD) method in cylindrical coordinates.

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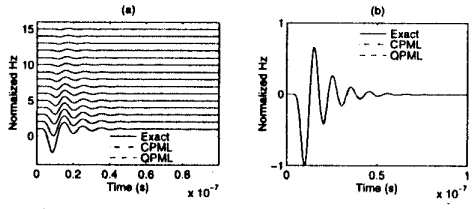

Figure 1. Comparison between analytical and numerical results for CPML and QPML schemes. (a) Waveforms for a set of receivers in the axisymmtric case. (b) Waveform at the fourth receiver.

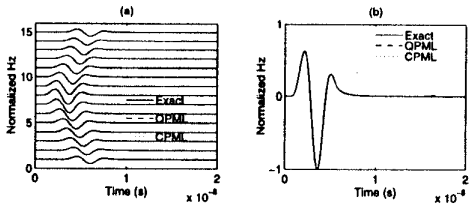

Figure 2. (a) Comparison between analytical and numerical results for CPML and QPML schemes in the homogeneous polar case. (b) The waveform of the eighth receiver.

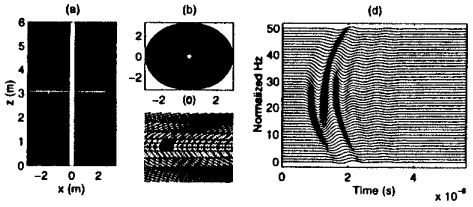

Figure 3. (a), (b), and (c) The geometry of the borehole radar probing application. (d) The scattered waveforms of the two fractures.