AN EFFICIENT PSTD ALGORITHM IN CYLINDRICAL COORDINATES

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I. Introduction

The finite-difference time-domain (FDTD) method has been enjoying its widespread applications in the simulations of transient electromagnetic wave propagation and scattering since it was first proposed by Yee in 1966 [1]. However, as the available computer memory and computational speed grow rapidly so that unprecedented large-scale problems can be solved, the FDTD method starts to show its limitation because of its relative large phase dispersion error. As the problem size increases, so does the required number of unknowns per wavelength.

In cylindrical coordinates, the conventional FDTD method encounters yet two more difficulties: (i) the requirement for a very small Δt because of the high concentration of cells near the z axis, and (ii) the singularity at the cylindrical axis. Although various remedies have been proposed, the treatment is not straightforward, and requires extra manipulations and computation time. In this work we propose a pseudospectral time-domain (PSTD) method for 3-D cylindrical and 2-D polar coordinates which will overcome these difficulties and significantly increase the efficiency.

II. Formulation

The cylindrical PSTD algorithm uses the formulations of the quasi-PML [3, 6] and true PML [2, 6] in cylindrical coordinates. For example, with the quasi-PML, the time-domain split Maxwell's equations can be written as [6]

$$a_\rho\epsilon\frac{\partial E_\rho^{(\phi)}}{\partial t} + (\omega_\rho\epsilon + a_\rho\sigma)E_\rho^{(\phi)} + \omega_\rho\sigma\int_{-\infty}^{t} E_\rho^{(\phi)}(\tau)d\tau = \frac{1}{\rho}\frac{\partial H_z}{\partial\phi} - J_\rho^{(\phi)}, \tag{1}$$

$$a_z\epsilon\frac{\partial E_\rho^{(z)}}{\partial t} + (\omega_z\epsilon + a_z\sigma)E_\rho^{(z)} + \omega_z\sigma\int_{-\infty}^{t} E_\rho^{(z)}(\tau)d\tau = -\frac{\partial H_\phi}{\partial z} - J_\rho^{(z)}, \tag{2}$$

$$a_\rho\epsilon\frac{\partial E_\phi^{(\rho)}}{\partial t} + (\omega_\rho\epsilon + a_\rho\sigma)E_\phi^{(\rho)} + \omega_\rho\sigma\int_{-\infty}^{t} E_\phi^{(\rho)}(\tau)d\tau = -\frac{\partial H_z}{\partial\rho} - J_\phi^{(\rho)}, \tag{3}$$

$$a_z\epsilon\frac{\partial E_\phi^{(z)}}{\partial t} + (\omega_z\epsilon + a_z\sigma)E_\phi^{(z)} + \omega_z\sigma\int_{-\infty}^{t} E_\phi^{(z)}(\tau)d\tau = \frac{\partial H_\rho}{\partial z} - J_\phi^{(z)}, \tag{4}$$

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$$a_\rho \epsilon \frac{\partial E_z}{\partial t} + (\omega_\rho + a_\rho \sigma) E_z + \omega_\rho \sigma \int_{-\infty}^{t} E_z(\tau) d\tau = \frac{1}{\rho} \frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} - J_z, \tag{5}$$

In the above, the split field components are $E_\rho = E_\rho^{(\phi)} + E_\rho^{(z)}$, and $E_\phi = E_\phi^{(\rho)} + E_\phi^{(z)}$. The other set of equations for updating $\mathbf{H}$ can be obtained by duality. Note that in the quasi-PML formulation, there is no need to split E_z and H_z. Therefore, the total number of unknown field components is 10 instead of 12 as for the true PML [6], which is omitted here for brevity.

The PSTD algorithm uses a centered grid where all field components are located at the center of each cell. Therefore, the first benefit of this algorithm is the removal of the singularity at cylindrical axis present in the staggered grid.

In the PSTD algorithm, the treatment in z derivative is exactly the same as in Cartesian coordinates [5]. For the spatial derivative $\partial/\partial\phi$, it is easily obtained by FFT since there is a natural periodicity in the ϕ direction.

The treatment of the ρ derivative is more complicated compared to the Cartesian coordinates, simply because that the boundary at $\rho = 0$ is not an open boundary. One way of treating this is to use Chebyshev pseudospectral method which inevitably increases the number of nodes at $\rho = 0$ and has a stringent stability criterion for Δt. Below we present two ways to use the Fourier series for ρ derivatives.

(a) The asymmetric form of PSTD algorithm in ρ direction

The most straightforward way to approximate the ρ derivative $\partial f(j_\rho, j_\phi)/\partial\rho$ is

$$\frac{\partial f}{\partial \rho} \approx \mathcal{D}_\rho f = \mathcal{F}_\rho^{-1}\Big\{ik_\rho \mathcal{F}_\rho[f]\Big\}, \tag{6}$$

where $\mathcal{F}_\rho$ and $\mathcal{F}_\rho^{-1}$ denote the forward and inverse FFT in ρ direction. Since $\rho = 0$ is a physical boundary, PML cells have to be placed near the outer boundary $\rho = \rho_{max}$ to remove the wraparound effect due to the periodicity of the DFT.

There are two major disadvantages associated with this approach: (i) More PML cells (usually around 20) are required near the outer boundary since the periodicity applies here (in contrast to a perfect electric conductor for the FDTD). (ii) Because of the periodicity, the negligibly small field at ρ_{max} (due to the PML attenuation) imposes a null-field condition at $\rho = 0$, effectively creating a small ghost source at the cylindrical axis. As observed from numerical experiments, this ghost source, although small, produces noticeable spurious fields.

(b) The symmetric form of PSTD algorithm in ρ direction

A much better way to treat the ρ derivatives is to use the symmetric form by assigning a new function for $0 \le j_\phi \le N_\phi/2 - 1$ (assuming N_ϕ is even) such that

$$g(j', j_\phi) = \begin{cases} f(-j'-1, j_\phi + N_\phi/2), & \text{for } j' = -N_\rho, \cdots, -1 \\ f(j', j_\phi), & \text{for } j' = 0, \cdots, N_\rho - 1 \end{cases} \tag{7}$$

Then the derivative is found by the FFT of these $N_\phi/2$ new arrays of length $2N_\rho$, in a way similar to (6). The total computation burden is reduced from (a) because only half the PML cells are needed. With this approach, both disadvantages in (a) have been removed.

III. Numerical Results

Figure 1 shows an example of a line source in a 2-D free space (polar coordinates). The source has a Blackman-Harris window time function with center frequency $f_c = 300$ MHz, and is located at $\rho_s = 1.5$ m, $\phi_s = 87.19°$. The computational domain is meshed by $N_\rho \times N_\phi = 32 \times 64$ cells with $\Delta\rho = 0.2$ m (or about 2 cells per wavelength at the frequency $2.5f_c$) and $\Delta t = 12.5$ ps. The snapshots show the effectiveness of the 10-layer PML ABC, while the last sub-plot shows the excellent agreement between the PSTD result and the analytical solution.

For the PSTD code to solve this problem on a SUN Ultra 1 workstation, it takes 140 seconds for the required 4000 time steps. For an acceptable accuracy, the FDTD method needs $N_\rho \times N_\phi = 128 \times 256$ cells, requiring 16 times more computer memory. In addition, a much smaller time step $\Delta t = 1.25$ ps has to be chosen for stability, requiring a total 40,000 time steps for the same problem. As a result, the FDTD code takes about 7 hours CPU time to complete this problem, or roughly 180 times slower.

We simulate the same source in an even larger problem. The center of source is located at $(\rho, \phi) = (35, 128)$ cells in a computational domain of $N_\rho \times N_\phi = 64 \times 256$ cells ($\rho_{max} = 10$ m). Fifteen receivers are set uniformly around a circle 30 cells away from the origin, and are 16 cells apart in ϕ direction. The first receiver is located at (30, 16). The numerical results agree well with analytical solutions, as shown in Figure 2.

Conclusions

The cylindrical pseudospectral time-domain method provides high accuracy and efficiency for time-domain simulations of waves in cylindrical and polar coordinates. It is ideal for large-scale problems since, in addition to the advantages in Cartesian coordinates, it provides two important advantages for cylindrical coordinates:

(1) The singularity at $\rho = 0$ is no longer present.

(2) The required number of time steps is reduced from FDTD by a factor of K^2, where K is the ratio of $\Delta\rho$ in PSTD and in FDTD. For the example shown, $K = 4$; It increases with the electrical size of the problem.

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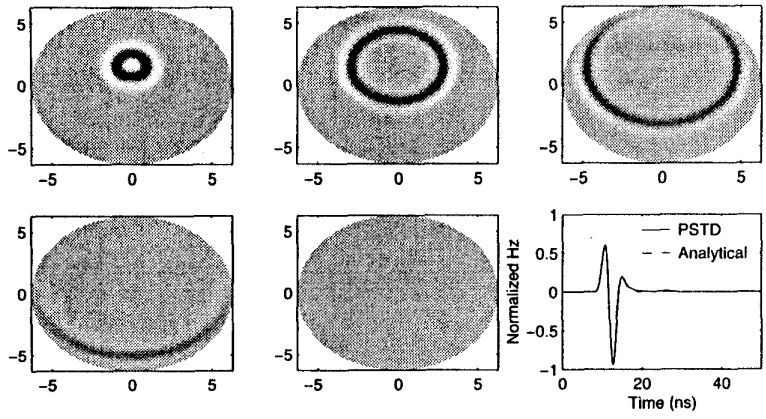

'igure 1. From first to fifth sub-plots, snapshots at time steps $n = 500, 1000, 1500, 2000,$ nd 2500 ($\Delta t = 12.5$ ps). The last plot compares the PSTD result with the analytical ›lution at $\rho = 3.1$ m, $\phi = 154.69°$. The source is located at $\rho_s = 1.5$ m, $\phi_s = 87.19°$.

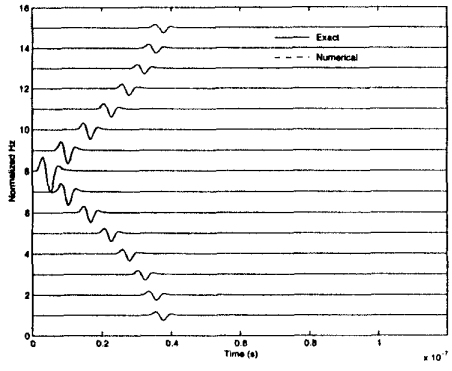

Figure 2. An array waveforms in free space.